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Generic Banach spaces and generic simplexes

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Abstract

We give a systematic study of certain class of generic Banach spaces. We show that they distinguish between an array of different properties related to smoothness of equivalent norms such as for example the Mazur intersection property or the existence of convex sets supported by all of their points. We also examine the dual constructions of generic Choquet simplexes with extra requirements such as for example those of Poulsen and Bauer asking that the set of extremal points is dense or closed, respectively.

Keywords: Generic spaces; Mazur intersection property; Support sets; Biorthogonal sequences

1. Introduction

Building a normed space or an infinite-dimensional simplex via finite-dimensional approximations is a natural idea already explored in several places in the literature (see, for example, [23,20,21,14,33,25]). For example, it is known that for a separable Banach space X, there is a $\subseteq$-directed family $\mathcal{F}$ of finite-dimensional subspaces of X isometric to corresponding ℓ_∞^n such that the union $\bigcup \mathcal{F} = \bigcup_{F\in\mathcal{F}} F$ is dense in X if and only if its dual X^* is isometric to a space of

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the form $L_1(\mu)$[1] if and only if X is an $\mathcal{L}_{\infty,\lambda}$-space for every $\lambda > 1$.[2] Recall also that X is called a *Gurarij space* if for every finite-dimensional normed spaces $E \subseteq F$, every isometry $T : E \to X$ and every $\varepsilon > 0$ there is an extension $U : F \to X$ of T such that $\|U\|, \|U^{-1}\| \leqslant 1 + \varepsilon$. Clearly, every Gurarij space is Lindenstrauss but not vice versa since all separable infinite-dimensional Gurarij spaces are pairwise isometric (see [14] and [27]) while the class of separable Lindenstrauss spaces contains a rich array of non-isomorphic spaces. The purpose of this paper is to lift these ideas to the level of set-theoretic forcing and give a systematic analysis of properties of the corresponding spaces. Our basic generic construction produces an Asplund space X_H, the strong differentiability space in which every continuous convex function is differentiable on a dense set of points in its domain. The Asplund space X_H is a c_0-saturated predual of $\ell_1(\omega_1)$. Simultaneously, for almost all examples we provide of such space X_H, we produce another generic space X which is a non-separable Gurarij space, and which is related to X_H via the following diagram

$$\begin{tikzcd} X \arrow[r, two heads, "\pi"] \arrow[dr, "q"'] & X_H \\ & X/\mathfrak{G} \arrow[u, "\equiv", "\bar{\pi}"'] \end{tikzcd} \tag{1}$$

where $\mathfrak{G}$ is the unique separable Gurarij space. Thus, X and X_H will share many common properties, so we arrive at the striking phenomenon that our Gurarij spaces will have an array of different structural properties and will therefore be non-isomorphic. Our main interest is however the properties of the Asplund space X_H having in mind the problems in the literature about the differentiability in the context of Banach spaces. We will achieve this by studying an array of properties related to the Mazur intersection property. Recall that a Banach space X has the Mazur intersection property if every closed convex set is the intersection of balls of X. That this is equivalent to smoothness of the norm of X is an important contribution of Giles, Gregory and Sims [11]. We shall however rely on another important result from this area given in [16] and which uses the characterization of [11] to show that the Mazur intersection property is closely related to the existence of biorthogonal systems on X. Another convexity problem in Banach spaces that can be reformulated as a problem about biorthogonal systems and therefore subject to our analysis here is an old problem of Rolewicz about Banach spaces X admitting convex subsets C supported[3] by all of their points (see [35]). For example, we shall construct generic Asplund and Gurarij spaces with or without the Mazur intersection property relative to any equivalent renorming and with or without the support sets. In fact we describe here many other examples of generic Banach spaces distinguishing essentially between any pair of biorthogonality properties considered in [12] solving thus an array of problems from that paper. For example, we show that for each rational $\varepsilon > 0$ there exist Gurarij and Asplund spaces X_ε and $X_{\varepsilon H}$ of density ω_1 which have uncountable ε-biorthogonal systems but no uncountable δ-biorthogonal systems for $0 \leqslant \delta < \varepsilon$. However, not all spaces that we construct here belong to one of these two classes. For example, we construct a Banach space X of density ω_1 with a normalized Schauder basis with constant $K > 1$ with no uncountable normalized basic sequence with basis constant $1 \leqslant K' < K$.

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1 Spaces with this property are called *Lindenstrauss spaces* in the literature.

2 That is, for every $\lambda > 1$ and every finite-dimensional subspace F of X there is a finite-dimensional $F \subseteq G \subseteq X$ λ-isomorphic to $\ell_\infty^{\dim(G)}$.

3 Recall that a C is supported by $x \in C$ if there is $f \in X^*$ such that $f(x) = \min_{y \in C} f(y) < \max_{y \in C} f(y)$.

Similarly we construct a generic Banach space X that has an uncountable $(1+\varepsilon)$-basic sequence for every $\varepsilon > 0$ but without uncountable monotone basic sequence. We don't know if such an example could exist in the classes of Lindenstrauss or Gurarij spaces.

As indicated by the title, the paper is also concerned with dual constructions, the generic constructions of *Choquet simplexes*.[4] When put in the proper context of set-theoretic forcing, one obtains a powerful projective limit construction of such simplexes. Not surprisingly constructions of similar kind have already appeared in the literature although only in the context of metrizable simplexes. The most striking such construction is the construction of a *Poulsen simplex*, an infinite-dimensional metrizable simplex $\mathbf{S}$ such that $\overline{\mathrm{Ext}(\mathbf{S})} = \mathbf{S}$ (see [33]). While the construction offered a considerable amount of freedom it turns out that $\mathbf{S}$ is unique up to affine homeomorphisms and that moreover $\mathbf{S}$ is a homogeneous and universal object in the class of metrizable simplexes (see [25]). We shall add to the theory of Poulsen simplexes by constructing a generic simplex $\mathbf{S}_{\omega_1}$ of weight ω_1 with a dense set of extreme points such that the space $P(\mathbf{S}_{\omega_1})$ of probability Radon measures equipped with the w^* topology is hereditarily separable in all finite powers. We shall see that a variation of this construction will give us a non-metrizable Poulsen simplex which is both hereditarily separable and hereditarily Lindelöf. This gives a solution to an old problem from the theory of Choquet simplexes asked in [29]. The perfect version of the simplex $\mathbf{S}_{\omega_1}$ also serves as a striking counterexample to Fremlin's problem about 2-to-1 pre-images of compact metric spaces (see, for example, [13]) since it is easily seen that every continuous map from $\mathbf{S}_{\omega_1}$ into a compact metric space must be constant on a non-metrizable subset of $\mathbf{S}_{\omega_1}$. We also construct a non-metrizable simplex $\mathbf{B}_{\omega_1}$ with $P(\mathbf{B}_{\omega_1})$ hereditarily separable which has the property that $\mathrm{Ext}(\mathbf{B}_{\omega_1})$ is a closed nowhere dense subset of $\mathbf{B}_{\omega_1}$, i.e., a simplex which in the literature is usually called a *Bauer simplex*. Having in mind the standard representation of Bauer simplexes as simplexes of the form $P(K)$ for compact Hausdorff K, this simplex can be viewed as the convex analogue of the generic zero-dimensional compactum $\mathbf{K}_0$ of [2] and in fact our construction of the perfect non-metrizable simplex mentioned above depends on the same idea.

We continue this Introduction with few remarks about the proof techniques of this paper which we find to be of independent interest. It turns out that most of the properties of generic Banach spaces or simplexes are proved by amalgamating finitely many finite-dimensional normed spaces or simplexes, respectively. We shall express this phenomenon using the notion that the certain norm *configurations* are *unavoidable*. While this will be made precise in Section 3.4 below, the intuitive meaning is that every uncountable ε-separated sequence of vectors (or an uncountable sequence of n-tuples of such vectors) contains a finite subsequence realizing the given norm configuration up to a given error. For example, already in the paper [37], which served as one of the sources of our inspiration, one finds a finite-dimensional amalgamation which can be used to show that configurations of the form

$$\left\| v_0 - \frac{1}{n}\sum_{i=1}^{n} v_i \right\| \leqslant \frac{1}{n} \max_{i\leqslant n} \|v_i\| \tag{2}$$

are unavoidable in many of our generic Banach spaces. Note that this in particular shows that these generic spaces contain no uncountable almost-biorthogonal systems and therefore cannot

[4] Recall that a compact convex subset K of locally convex topological vector space X is a Choquet simplex if it has the property that the cone $C = \{(\lambda x, \lambda)\colon x \in K,\ \lambda \in \mathbb{R}\}$ defines a lattice order on $C - C \subseteq X \times \mathbb{R}$.

be renormed to have the Mazur intersection property. On the dual side, in the paper [2] appeared three years earlier, and that served as our other source of inspiration, one finds the following configuration[5] that is in some sense dual to (2)

$$\bigcup_{i=1}^{n}[s_i^0]\subseteq\bigcap_{k=1}^{n+1}\bigcup_{i=1}^{n}[s_i^k]\quad\text{and}\quad\prod_{i=1}^{n}[s_i^0]\subseteq\bigcup_{k=1}^{n+1}\prod_{i=1}^{n}[s_i^k]\tag{3}$$

with a proof that its unavoidability implies that the corresponding exponential space $\exp(K_0)$, the space of nonempty closed subsets of K_0 equipped with the Vietoris topology, is hereditarily separable. We shall see that a slight sharpening of this configuration will yield the stronger conclusion that the space $P(K_0)$ of probability measures on $\mathbf{K}_0$ (and many of the other generic simplexes mentioned above) is hereditarily separable. The paper [4] shows that other topological properties of these two functors can also be captured by finite configurations. Treating the subtle difference between the basis constants or the measures of biorthogonality has led us to the proof that the configurations of the form[6]

$$\left\|(v_0-v_1)-\frac{1}{m}\sum_{i=1}^{n}(v_{2i}-v_{2i+1})\right\|<\frac{\delta}{2}\quad\text{and}\quad\left\|\sum_{i=1}^{n}v_i\right\|\leqslant C\left\|\sum_{i=1}^{n}v_i-\sum_{i=n+1}^{2n}v_i\right\|\tag{4}$$

are unavoidable in the corresponding generic spaces. It turns out that there is also a norm configuration whose unavoidability implies that there is no closed convex set supported by all of its points. More precisely, we solve Rolewicz's problem by a finite-dimensional amalgamation which shows that the norm configuration

$$\left\|-\sum_{i=0}^{k(m+1)-1}v_i+k\cdot v_{k(m+1)}+\sum_{i=k(m+1)+1}^{k(2m+1)}v_i\right\|\leqslant 2\max_{i\leqslant k(2m+1)}\|v_i\|\tag{5}$$

is unavoidable for suitably chosen integers k and m. This works equally well for the function space $C(\mathbf{K}_0)$ over the generic compactum $\mathbf{K}_0$ of [2] showing that the function space $C(K_0)$ does not contain a support set, a result originally announced in [39]. We note that another generic compactum K whose function space $C(K)$ admits no support set was independently constructed in [17]. Our paper contains many more examples of unavoidable configurations and in fact we reach limits of our finite-dimensional amalgamation techniques when we show that a basis of an arbitrary finite-dimensional normed space is block representable inside an arbitrary uncountable ε-separated sequence of vectors. We feel however that we have only barely touched a rich subject of finite-dimensional norm amalgamation techniques that will lead to many more interesting examples of non-separable as well as separable Banach spaces.

We finish the Introduction with a list of specific Banach spaces and simplexes constructed in this paper. In this list we use $(X;Y)$ to denote a *pair* of non-separable Banach spaces X and Y such that X is Gurarij, Y Asplund and c_s0-saturated, Y^* is isometric to $\ell_1(\omega_1)$ and such that Y is isometric to a quotient of X by the separable Gurarij space (see the diagram (1) above).

[5] Here $[s]$ denotes the basic open subset of the generic compactum $\mathbf{K}_0\subseteq\{0,1\}^{\omega_1}$ determined by a finite partial function s from ω_1 into $\{0,1\}$. The duality that we refer to here is between the hereditary separability properties of the space $P(K_0)$ of Radon probability measures on K_0 and the hereditary Lindelöf properties of the weak topology of $C(K_0)$.

[6] Of course, under some natural conditions on the constants involved.

- A pair of spaces $(X; Y)$ such that both X and Y equipped with their weak topologies are hereditarily Lindelöf in all finite powers, but no equivalent dual ball of X or Y is weak*-sequentially separable. Moreover X and Y contain no support sets.
- For each $\varepsilon > 0$ a pair $(X; Y)$ with ε-biorthogonal sequences but no uncountable η-biorthogonal sequence for any $\eta < \varepsilon/(1+\varepsilon)$. Moreover none of these spaces have support sets nor uncountable ω-independent sequences.
- A polyhedral Banach space whose norm depends on finitely many coordinates without the Mazur Intersection Property, and without uncountable biorthogonal systems.
- A pair $(X; Y)$ such that X and Y have uncountable ε-biorthogonal sequence for every $\varepsilon > 0$ but neither X nor Y have uncountable biorthogonal sequences nor do they admit support sets.
- A Banach space X with a normalized Schauder K-basis of length ω_1 but with no uncountable K'-basic sequence for any $1 \leqslant K' < K$.
- A Banach space X with a Schauder basis which has an uncountable $(1+\varepsilon)$-basic sequence for every $\varepsilon > 0$ but with no uncountable monotone basic sequences.
- Any separated and normalized sequence $(x_\alpha)_{\alpha<\omega_1}$ of vectors of any of the above pairs of spaces $(1+\varepsilon)$-block-represents any finite basic sequence in any other Banach space.
- If E is any of the above spaces and if Z is a subspace of E, then every bounded operator $T : Z \to E$ is of the form $T = \lambda \cdot i + S$ where $\lambda \in \mathbb{R}$, $i : Z \to E$ is the inclusion map and S has separable range.
- If E is any of the above spaces and if $T : E \to F$ is a quotient operator, then either $\operatorname{Ker} T$ is separable or F is separable.
- There is a Poulsen simplex K of weight $\aleph_1$ such that its space $P(K)$ of probability measures is hereditarily separable in all finite powers.
- There is a Bauer simplex K of weight $\aleph_1$ such that its space $P(K)$ of probability measures is hereditarily separable in all finite powers.
- The extremal points of the Bauer simplex is a zero-dimensional compactum K_0 of weight $\aleph_1$ whose function space $C(K_0)$ contains no support sets.
- There is a perfect Poulsen simplex K whose space $P(K)$ of probability measures is hereditarily separable in all finite powers.

Moreover, we show that a single forcing extension of the universe of sets will have all these objects. In fact, in the forthcoming article [26] we proceed differently and show that the set-theoretic principle $\diamondsuit$ implies the existence of all these examples of spaces and simplexes. It should be noted that prior to our constructions, besides the generic $C(K)$-spaces admitting no support sets mentioned above, the only previously known related examples are those of Kunen (appearing in [31]) and the second author [38] of a non-separable Asplund $C(K)$-space with no uncountable ε-biorthogonal sequences and the example of Shelah [37] of a non-separable Gurarij space with no uncountable ε-biorthogonal sequences. The construction in [31] uses CH while the construction of [38] uses considerably less, the cardinal equality $\mathfrak{b} = \omega_1$. The construction of [37] uses $\diamondsuit$. It should also be noted that some additional principles are in fact necessary since it is known (see [40] and [1]) that the strong Baire category assumptions imply that an arbitrary non-separable Banach space admits an uncountable biorthogonal system and that an arbitrary Asplund space of density $\aleph_1$ has an equivalent renorming with the Mazur intersection property.

2. Preliminaries and notation

We use the standard terminology from [24]. In our constructions the following spaces play a central role.

Definition. Let X be a Banach space.

(a) X is an $\mathcal{L}_{\infty,\lambda}$-space ($\lambda \geqslant 1$) if for every finite-dimensional subspace F of X there is a finite-dimensional subspace $F \subseteq G \subseteq X$ such that $d(G, \ell_\infty^{\dim G}) \leqslant \lambda$, where $d(G, \ell_\infty^{\dim G})$ is the Banach–Mazur distance defined by $\inf\{\|T\| \cdot \|T^{-1}\|\colon T : G \to \ell_\infty^{\dim G}$ is an isomorphism$\}$.
(b) X is a *Lindenstrauss space* (or an L_1-predual) when X^* is isometric to $L_1(\mu)$ for some measure μ.
(c) X is a *Gurarij space* if for every finite-dimensional normed spaces $E \subseteq F$, every isometry $T : E \to X$ and every $\varepsilon > 0$ there is an extension $U : F \to X$ of T such that $\|U\|, \|U^{-1}\| \leqslant 1 + \varepsilon$.

The following result connects the previous notions.

Theorem. *Let X be a separable infinite-dimensional Banach space. Then the following are equivalent*:

(a) *X is a Lindenstrauss space.*
(b) *X is $\mathcal{L}_{\infty,\lambda}$ for every $\lambda > 1$.*
(c) *X is a π_1^∞-space, i.e. there is a $\subseteq$-directed family $\mathcal{F}$ of finite-dimensional subspaces of X isometric to corresponding ℓ_∞^n such that the union $\bigcup \mathcal{F} = \bigcup_{F \in \mathcal{F}} F$ is dense in X.*

The equivalence of (a) *and* (b) *and the implication* (c) $\to$ (a) *is true for arbitrary Banach space.*

For an arbitrary Banach space the implication (a) $\to$ (b) was proved by Lazar and Lindenstrauss in [20, Theorem 1, p. 205], and the reverse one (b) $\to$ (a) was proved for arbitrary Banach space by Lindenstrauss in [23, Corollary 1, p. 66]. A not so difficult approximation argument proves that (c) implies (b) for arbitrary Banach space. Finally, the implication (b) $\to$ (c) is a result of Michael and Pelczynski [30, Theorem 1.1, p. 190].

It follows then that Gurarij spaces are always Lindenstrauss spaces. A remarkable property of separable Gurarij spaces proved by W. Lusky [27] is that they are all *isometric*. Let us denote the unique separable Gurarij space by $\mathfrak{G}$. In the non-separable context Lusky [28] gave examples of non-separable Gurarij spaces of density $\geqslant 2^{\aleph_0}$ that are not isometric, not even isomorphic. We shall need the following well-known characterization of the corresponding dual spaces.

Theorem. *(See [22].) Suppose that X is a separable $\mathcal{L}_\infty$ space. Then its dual space X^* is isomorphic, either to ℓ_1 or to $M[0,1]$, the space of Radon measures on the unit interval. Moreover, this last case $X^* \cong M[0,1]$ only happens when ℓ_1 embeds isomorphically in X.*

Finally, we introduce less standard terminology and notions.

For $0 \in A \subseteq \mathbb{R}$ and I an arbitrary set, we define $c_{00}(I, A)$ as the collection of all $f : I \to A$ with $\operatorname{supp} f = \{i \in I\colon f(i) \neq 0\}$ finite. In case that A is a field, $c_{00}(I, A)$ is a vector space. For each $i \in I$, let u_i be the vector of $c_{00}(I, \mathbb{R})$ defined by $u_i(j) = \delta_{i,j}$. Given $J \subseteq I$ and

$f \in c_{00}(I, A)$ we denote by $f \upharpoonright J = \sum_{j\in J} f(j)u_j$ the restriction of f to J, and in general given $F \subseteq c_{00}(I)$ and $J \subseteq I$, we denote by $F \upharpoonright J = \{f \upharpoonright J\colon f \in F\}$ the set of restrictions of elements of F to J.

For $x \in c_{00}(I, A)$ and $i \in I$, we write $(x)_i$ to denote the ith-coordinate of x. We define two operations $\vee$ and $\wedge$ in $c_{00}(I, \mathbb{R})$ as follows: For $x, y \in c_{00}(I, \mathbb{R})$, let

$$(x \vee y)_\alpha := \begin{cases} (x)_\alpha & \text{if } |(x)_\alpha| \geqslant |(y)_\alpha|, \\ (y)_\alpha & \text{otherwise,} \end{cases}$$
$$(x \wedge y)_\alpha := \begin{cases} (y)_\alpha & \text{if } |(x)_\alpha| \geqslant |(y)_\alpha|, \\ (x)_\alpha & \text{otherwise} \end{cases}$$

for every $\alpha \in I$.

Recall that ℓ_∞^n is the normed space $(\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|(x_i)_{i<n}\|_\infty = \max_{i<n} |x_i|$, or equivalently, whose unit cell is the n-dimensional cube. The following notion is the key in our constructions.

Definition 2.1. A *$\mathbb{Q}$-f.d. space* H is an isometrical copy of the $\mathbb{R}$-span of a $\mathbb{Q}$-subspace of the $\mathbb{Q}$-vector space ℓ_∞^n endowed with the sup-norm.

The main examples of $\mathbb{Q}$-f.d. spaces are the isometric copies of ℓ_∞^n's. Notice that a subspace H of some ℓ_∞^n is a $\mathbb{Q}$-f.d. space if an only if H is the real span of a $\mathbb{Q}$-subspace of ℓ_∞^n. In particular, each ℓ_1^n is also a $\mathbb{Q}$-f.d. space.

By definition, if H is a $\mathbb{Q}$-f.d. space, then the unit cells of H and of H^* are *polyhedrons*, i.e. they have finitely many extremal points. Moreover, H has a basis $(x_i)_{i<n}$ such that the evaluation of each x_i in an extremal point of B_{H^*} is a rational number. It is not difficult to see that this is characterization of $\mathbb{Q}$-f.d. spaces. We call a such basis of H a *$\mathbb{Q}$-basis*. We call a point $x \in H$ a *$\mathbb{Q}$-point* if x is a rational linear combination of an (any) $\mathbb{Q}$-basis of H, and $(x_i)_{i<k}$ is a *$\mathbb{Q}$-sequence* of H if it is a sequence of $\mathbb{Q}$-points. Finally, we say that $(x_i)_{i<k}$ is a $\mathbb{Q}$-sequence (basis) if it is a $\mathbb{Q}$-sequence (basis) of some $\mathbb{Q}$-f.d. space H.

We list now few more of the basic facts about $\mathbb{Q}$-f.d. spaces which we freely use below.

Proposition 2.2.

1. *H is a $\mathbb{Q}$-f.d space if and only if H^* is a $\mathbb{Q}$-f.d. space. If H is a $\mathbb{Q}$-f.d. space, then we call the elements of H^* as $\mathbb{Q}$-functionals.*
2. *If $(x_i)_{i<n}$ and $(y_i)_{i<n}$ are two $\mathbb{Q}$-bases then one is a $\mathbb{Q}$-linear combination of the other.*
3. *If H_0 is a $\mathbb{Q}$-f.d. space and H_1 is a subspace of H_0, then H_1 is a $\mathbb{Q}$-f.d. space if and only if H_1 is the span of $\mathbb{Q}$-points of H_0.*
4. *For every finite-dimensional normed space F and every $\varepsilon > 0$ there is a $\mathbb{Q}$-f.d. space H such that $d(F, H) \leqslant \varepsilon$, where d is the Banach–Mazur distance.*

Suppose now that $H_1 \subseteq H_0$ are two $\mathbb{Q}$-f.d. spaces.

5. *Every $\mathbb{Q}$-functional of H_1 can be extended to a $\mathbb{Q}$-functional of H_0.*
6. *If H_1 is a proper subspace of H_0, then there is a normalized $\mathbb{Q}$-functional f of H_0 such that $f \upharpoonright H_1 = 0$.*

7. *If $x \in H_0$ is a $\mathbb{Q}$-vector then there is a $\mathbb{Q}$-functional f of H_0 such that $d(x, H_1) = f(x)$ and $f \restriction H_1 = 0$.*
8. *There is a $\mathbb{Q}$-f.d. subspace H_2 of H_0 which is a complement of H_1 in H_0.* $\square$

Definition 2.3. A $\mathbb{Q}$-isometry between two $\mathbb{Q}$-f.d. spaces H_0 and H_1 is an isometry $T : H_0 \to H_1$ such that if $(x_i)_{i<m}$ and $(y_i)_{i<n}$ are (any) $\mathbb{Q}$-bases of H_0 and H_1 then $T(x_i)$ is in the $\mathbb{Q}$-span of $(y_i)_{i<n}$.

It is clear that the composition of two $\mathbb{Q}$-isometries is again a $\mathbb{Q}$-isometry.

3. Generic normed spaces

When we say that a Banach space X of density ω_1 is *generic* we have in mind a particular context, or more precisely a set $\mathbb{P}$ of finite approximations to a norm $\|\cdot\|$ on $c_{00}(\omega, \mathbb{Q})$ which when completed would give us a Banach space of density ω_1. The set $\mathbb{P}$ is sometimes called *the forcing notion* and the elements of $\mathbb{P}$ are sometimes called *the conditions* on the resulting norm and the following lists some of the most natural properties of $\mathbb{P}$.

Definition 3.1 *(Basic forcing notion).* Let $\mathbb{P}_{\mathrm{basic}}$ be the set of $p = (D_p, F_p, A_p, H_p)$, called *conditions*, with the following properties:

(C.1) $A_p \subseteq D_p \subseteq \omega_1$ are finite.
(C.2) $H_p \subseteq F_p \subseteq c_{00}(D_p, \mathbb{Q} \cap [-1, 1])$ are finite and F_p is symmetric.
(C.3) $H_p = \{h_\gamma^{(p)}\}_{\gamma \in D_p}$ is such that for every $\gamma \in D_p$ one has that

$$h_\gamma^{(p)} \restriction \gamma = 0 \quad \text{and} \quad \left(h_\gamma^{(p)}\right)_\gamma \neq 0. \tag{6}$$

The ordering $p \leqslant_{\mathrm{b}} q$ is defined by:

(O.1) $D_q \subseteq D_p$ and $A_q \cap D_p = A_p$.
(O.2) $F_q \subseteq F_p \restriction D_q = \{f \restriction D_q\colon f \in F_p\} \subseteq \mathrm{conv}_{\mathbb{Q}}(F_q)$.
(O.3) $H_q \subseteq H_p \restriction D_q \subseteq \mathrm{conv}_{\mathbb{Q}}(\pm H_q)$.

A *forcing notion* $\mathbb{P}$ is any subset of $\mathbb{P}_{\mathrm{basic}}$ partially ordered by $\leqslant_{\mathrm{b}}$. Given a forcing notion $\mathbb{P}$ and an ordinal $\alpha < \omega_1$ we define

$$\mathbb{P}_\alpha := \{p \in \mathbb{P}\colon D_p \subseteq \alpha\}$$

endowed with the basic ordering $\leqslant_{\mathrm{b}}$.

The *domain* $\Delta_{\mathbb{P}}$ of $\mathbb{P}$ is defined by

$$\Delta_{\mathbb{P}} := \left\{\left(h_\gamma^{(p)}\right)\colon \gamma \in A_p,\ p \in \mathbb{P}\right\}.$$

Finally, given $p \in \mathbb{P}$ and $\delta \in \Delta_{\mathbb{P}}$ we define $A_p^{(\delta)} := \{\gamma \in D_p\colon (h_\gamma^{(p)})_\gamma = \delta\}$.

Definition 3.2. For a given $p \in \mathbb{P}_{\mathrm{basic}}$, we can naturally define the following seminorms on the f.d. vector space $c_{00}(D_p)$. Given $x \in c_{00}(D_p)$ let

$$\|x\|_p := \max\{\langle f, x\rangle: f \in F_p\}, \tag{7}$$

$$\|x\|_{p,H} := \max\{|\langle h_\gamma^{(p)}, x\rangle|: \gamma \in D_p\}. \tag{8}$$

Remark 3.3.

(a) The condition (O.2) is equivalent to:
(O.2′) For every $f \in F_q$ there is $g \in F_p$ such that $g \upharpoonright D_q = f$ and for every $x \in c_{00}(D_q)$ one has that $\|x\|_p = \|x\|_q$. Similar equivalence is also true for condition (O.2).
(b) By (C.3) and (O.3), it follows that if $p \leqslant q$ and $\gamma \in D_q$ then

$$h_\gamma^{(p)} \upharpoonright D_q = h_\gamma^{(q)}. \tag{9}$$

We list some facts.

Proposition 3.4. *Let $p \in \mathbb{P}_{\mathrm{basic}}$.*

(a) *Both $\|\cdot\|_p$ and $\|\cdot\|_{p,H}$ are norms and $\|\cdot\|_p \geqslant \|\cdot\|_{p,H}$.*
(b) $\mathrm{conv}_{\mathbb{R}}(F_p) = B_{(X_p)^*}$ *and* $\mathrm{conv}_{\mathbb{R}}(\pm H_p) = B_{(X_{p,H})^*}$.
(c) $\mathrm{Ext}(B_{(X_p)^*}) \subseteq \pm F_p = F_p$ *and* $\mathrm{Ext}(B_{(X_{p,H})^*}) \subseteq \pm H_p$.
(d) *For every $f \in B_{(X_{p,H})^*}$ and every α there is a unique sequence $(a_\gamma)_{\gamma\in D_p\cap\alpha}$ such that*

$$f \upharpoonright D_p \cap \alpha = \sum_{\gamma\in D_p\cap\alpha} a_\gamma \cdot h_\gamma^{(p)} \upharpoonright \alpha.$$

Moreover, the sequence $(a_\gamma)_\gamma$ satisfies that $\sum_\gamma |a_\gamma| \leqslant 1$. And in particular,

$$\mathrm{Ext}\, B_{(X_{p,H})^*} = \pm H_p. \tag{10}$$

(e) *The linear mapping*

$$T : X_{p,H} \to \ell_\infty(D_p)$$
$$x \mapsto T(x) = \big(h_\gamma^{(p)}(x)\big)_{\gamma\in D_p}$$

is an isometry onto. So, $X_{p,H}$ is isometric to $\ell_\infty(D_p)$.
(f) *The two spaces X_p and $X_{p,H}$ are clearly $\mathbb{Q}$-f.d. (Definition* 2.1).

Proof. (a): $\|\cdot\|_p \geqslant \|\cdot\|_{p,H}$ is clearly true, (C.3) gives that $\|\cdot\|_{p,H}$ is a norm, hence $\|\cdot\|_p$ is also a norm. (b) is a consequence of the Hahn–Banach Theorem. (c) follows from (b), and the fact that F_p is symmetric.

(d): (C.3) gives that $(h_\gamma^{(p)} \upharpoonright \alpha)_{\gamma\in D_p\cap\alpha}$ is a linear basis of $c_{00}(D_p \cap \alpha)$, hence, if $f \in B_{(X_{p,H})^*}$, then there is a unique sequence $(a_\gamma)_{\gamma\in D_p\cap\alpha}$ such that $f \upharpoonright \alpha = \sum_{\gamma\in D_p\cap\alpha} a_\gamma h_\gamma^{(p)} \upharpoonright \alpha$. On the other side, from (b) it follows that $f = \sum_{\gamma \in D_p} b_\gamma h_\gamma^{(p)} + \sum_{\gamma \in D_p} c_\gamma(-h_\gamma^{(p)})$ with $b_\gamma, c_\gamma \geqslant 0$ and $\sum_{\gamma \in D_p}(b_\gamma + c_\gamma) = 1$. So,

$$f \upharpoonright \alpha = \sum_{\gamma \in D_p \cap \alpha} (b_\gamma - c_\gamma) h_\gamma^{(p)} \upharpoonright \alpha,$$

because $h_\gamma^{(p)} \upharpoonright \alpha = 0$ for $\gamma \geqslant \alpha$. Hence, $a_\gamma = b_\gamma - c_\gamma$, and

$$\sum_{\gamma \in D_p \cap \alpha} |a_\gamma| = \sum_{\gamma \in D_p \cap \alpha} |b_\gamma - c_\gamma| \leqslant \sum_{\gamma \in D_p \cap \alpha} b_\gamma + c_\gamma \leqslant 1.$$

(e): T is clearly an isometry; T is onto because the dimension of the input and target space is the same. □

We give now some examples of forcing notions.

Examples 3.5. (I) Given a forcing notion $\mathbb{P}$, and a limit ordinal α, the corresponding subset $\mathbb{P}_\alpha = \{p \in \mathbb{P}\colon D_p \subseteq \alpha\}$ is also a forcing notion.

(II) Let $\mathbb{P}$ be a forcing notion. For every $p \in \mathbb{P}$ define $p_H = (D_p, \pm H_p, A_p, H_p)$, which is also a basic condition, i.e. $p_H \in \mathbb{P}_{\mathrm{basic}}$. Then $\mathbb{P}_H := \{p_H\colon p \in \mathbb{P}\}$ is a forcing notion.

(III) Let $\mathbb{P}_{\mathrm{a}}$ be the forcing notion consisting on all the basic conditions $p = (D_p, F_p, A_p, H_p)$ such that

(a) $A_p = D_p$ and $F_p = \pm H_p$.
(b) For every $\gamma \in D_p$ one has that $h_\gamma^{(p)} := u_\gamma$.

It follows that

$$\left\| \sum_{\gamma \in D_p} a_\gamma u_\gamma \right\|_p = \max_{\gamma \in D_p} |a_\gamma|, \tag{11}$$

so, the sequence $(u_\gamma)_{\gamma \in D_p}$ is 1-equivalent to the unit basis of $\ell_\infty(D_p)$.

(IV) Similarly, we can define the forcing notion $\mathbb{P}_{\mathrm{b}}$ consisting on all the basic conditions $p = (D_p, F_p, A_p, H_p)$ where $A_p = D_p$, and $(u_\gamma)_{\gamma \in D_p}$ is 1-equivalent to the unit basis of $\ell_1(D_p)$.

The basic intuition behind the forcing notion $\mathbb{P}$ is that the desired Banach space is obtained by taking some direct limit of the spaces X_p and for this we need the following notion. A subset $\mathbb{F}$ of $\mathbb{P}$ is a *filter* if:

(F.1) For all $p, q \in \mathbb{F}$, there is $r \in \mathbb{F}$ such that $r \leqslant p$ and $r \leqslant q$.
(F.2) For all $p, q \in \mathbb{P}$, if $p \leqslant q$ and $p \in \mathbb{F}$ then $q \in \mathbb{F}$.

Definition 3.6. Let $\mathbb{F}$ be any filter of $\mathbb{P}$. We define $D_{\mathbb{F}} := \bigcup_{p \in \mathbb{F}} D_p$, $A_{\mathbb{F}} := \bigcup_{p \in \mathbb{F}} A_p$ and on the corresponding vector space $c_{00}(D_{\mathbb{F}})$ we define naturally two norms $\|\cdot\|_{\mathbb{F}}$ and $\|\cdot\|_{\mathbb{F},H}$ as follows. For $x \in c_{00}(D_{\mathbb{F}})$ let

$$\|x\|_{\mathbb{F}} := \|x\|_p,$$
$$\|x\|_{\mathbb{F},H} := \|x\|_{p,H}$$

where $p \in \mathbb{F}$ is such that $x \in X_p$. We denote by $X_{\mathbb{F}}$ and $X_{\mathbb{F},H}$ the corresponding completions. Let $\pi_{\mathbb{G}} : X_{\mathbb{G}} \to X_{\mathbb{G},H}$ be the formal identity mapping. This is clearly a bounded operator of norm 1. Indeed, under some reasonable assumption on $\mathbb{P}$, the operator $\widetilde{\pi}_{\mathbb{G}} : X_{\mathbb{G}}/\operatorname{Ker}\pi_{\mathbb{G}} \to X_{\mathbb{G},H}$ is an isometry onto (see Theorem 3.18).

For every $\alpha < \omega_1$ we define

$$X_{\mathbb{F}}^{(\alpha)} := \overline{\langle u_\gamma\rangle_{\gamma\in D_{\mathbb{F}}\cap\alpha}} \subseteq X_{\mathbb{F}},$$
$$X_{\mathbb{F},H}^{(\alpha)} := \overline{\langle u_\gamma\rangle_{\gamma\in D_{\mathbb{F}}\cap\alpha}} \subseteq X_{\mathbb{F},H}.$$

Define also

$$h_\gamma := h_\gamma^{(\mathbb{F})} := \bigvee_{p\in\mathbb{F}} h_\gamma^{(p)}, \quad \text{for } \gamma < \omega_1, \quad \text{and}$$
$$H_{\mathbb{F}} := \{h_\gamma\colon\ \gamma \in D_{\mathbb{F}}\}.$$

It is clear that $h_\gamma \in \ell_\infty(D_{\mathbb{F}})$ and, by (9), one has that

$$h_\gamma \restriction D_p = h_\gamma^{(p)} \quad \text{for every } p \in \mathbb{F}. \tag{12}$$

It follows then that $h_\gamma^{(\mathbb{F})}$ is a bounded linear functional of $c_{00}(D_{\mathbb{F}})$ of norm at most 1. Recall that for $x \in c_{00}(D_{\mathbb{F}})$, we write

$$h_\gamma(x) := \langle h_\gamma, x\rangle = \sum_{\alpha\in D_p} (h_\gamma)_\gamma \cdot (x)_\alpha.$$

It follows that $h_\gamma \in B_{(X_{\mathbb{F}})^*}$.

In general, for every $f \in B_{(X_{\mathbb{F}})^*}$ and every $p \in \mathbb{F}$ one has that $f \restriction D_p \in B_{(X_p)^*} = \mathrm{co}_{\mathbb{R}}(F_p)$, so we can make the following natural identification,

$$B_{(X_{\mathbb{F}})^*} = \bigl\{f \in \ell_\infty(D_{\mathbb{F}})\colon\ f \restriction D_p \in \mathrm{co}_{\mathbb{R}}(F_p) \text{ for every } p \in \mathbb{F}\bigr\}.$$

Then for each $f \in (X_{\mathbb{F}})^*$ one has that

$$\|f\|_\infty := \sup_{\gamma\in D_{\mathbb{F}}} \bigl|f(u_\gamma)\bigr| \leqslant \|f\|_{(X_{\mathbb{F}})^*}.$$

In other words, $f \in (X_{\mathbb{F}})^* \mapsto (f(u_\gamma))_{\gamma<\omega_1} \in \ell_\infty(\omega_1)$ is a norm 1 operator, and in addition it is 1–1.

We list some facts to be used freely later on.

Proposition 3.7. *Let $\mathbb{F}$ be a filter of $\mathbb{P}$. Then*:

(a) *For every $\gamma \in D_{\mathbb{F}}$ one has that $\|u_\gamma\|_{\mathbb{F}} \leqslant 1$.*
(b) *For every $x \in X_{\mathbb{F}}$ and every $\varepsilon > 0$ there is $\delta > 0$ such that if $f \in B_{(X_{\mathbb{F}})^*}$ is such that $\|f\|_\infty \leqslant \delta$ then $|f(x)| \leqslant \varepsilon$.*
(c) *Every uncountable sequence $(x_\alpha)_{\alpha<\omega_1}$ of different points of $c_{00}(D_{\mathbb{F}}, \mathbb{Q})$ contains an uncountable subsequence $(x_\alpha)_{\alpha\in I}$ which is separated, i.e., $\inf_{\alpha\neq\beta \text{ in } I} \|x_\alpha - x_\beta\| > 0$.*
(d) *For every uncountable sequence $(x_\gamma)_{\gamma\in\omega_1}$ of points of $X_{\mathbb{F}}$ and every $\varepsilon > 0$ there is $\delta > 0$ and an uncountable subsequence $(x_\gamma)_{\gamma\in\Gamma}$ such that for every $\gamma \in \Gamma$ and every $f \in B_{(X_{\mathbb{F}})^*}$ such that $\|f\|_\infty \leqslant \delta$ one has that*

$$\bigl|f(x_\gamma)\bigr| \leqslant \varepsilon.$$

(e) *The set $H_{\mathbb{F}}$ is 1-norming of $X_{\mathbb{F},H}$.*

Proof. (a) is trivial. (b): Let $y \in c_{00}(\mathbb{F})$ be such that $\|y - x\|_{\mathbb{F}} \leqslant \varepsilon/2$. Then $\delta = \varepsilon/(2 \cdot \|y\|_{\ell_1})$ does the job.

(c): For each $\alpha < \omega_1$, let $\xi_\alpha = \max \operatorname{supp} x_\alpha$. Since the points of the sequence are distinct, it follows that there is an uncountable subset $\Gamma \subseteq \omega_1$ and a positive rational number ε such that for $\alpha < \beta$ in Γ one has that $\xi_\alpha < \xi_\beta$ and $h_{\xi_\alpha}(x_\alpha) = \varepsilon$. It follows by the property (C.3) of the forcing $\mathbb{P}$ that $(x_\alpha)_{\alpha\in\Gamma}$ is ε-separated. (e) is trivial. □

Definition 3.8. Given a basic condition $p \in \mathbb{P}_{\text{basic}}$ and an ordinal $\alpha < \omega_1$ we define the basic condition $p \restriction \alpha = (D_p \restriction \alpha, F_p \restriction \alpha, H_p \restriction \alpha)$ as follows:

$$D_p \restriction \alpha = D_p \cap \alpha, \qquad F_p \restriction \alpha = \{f \restriction \alpha\colon f \in F_p\},$$
$$A_p \restriction \alpha = A_p \cap \alpha, \qquad H_p \restriction \alpha = \bigl\{h_\gamma^{(p)} \restriction \alpha\colon \gamma \in D_p \cap \gamma\bigr\}.$$

Let $\theta : A \to \omega_1$ be any order-preserving mapping. Given a basic condition $p \in \mathbb{P}_{\text{basic}}$ with $D_p \subseteq A$ we define the θ-*spread* of p as the basic condition $\theta(p) = (\theta(D_p), \theta(F_p), \theta(H_p))$ by

$$\theta(F_p) := \bigl\{\theta(f)\colon f \in F_p\bigr\},$$
$$\theta(H_p) := \bigl\{\theta\bigl(h_{\theta(\gamma)}^{(p)}\bigr)\colon \gamma \in D_p\bigr\},$$

where we are using that the order-preserving mapping θ naturally defines an isomorphism between $c_{00}(D_p)$ and $c_{00}(\theta(D_p))$.

Note that for every basic conditions $p \leqslant q$, every $\alpha < \omega_1$ and every order-preserving mapping $\theta : D_p \to \omega_1$, we have that $p \leqslant p \restriction \alpha \leqslant q \restriction \alpha$ and $\theta(p) \leqslant \theta(q)$.

3.1. Dense sets and generic filters

In proving local properties of our generic spaces the following notion of dense sets plays a crucial role.

Definition 3.9. A subset $\mathcal{D}$ of $\mathbb{P}$ is *dense* if for all $p \in \mathbb{P}$ there is $q \in \mathcal{D}$ such that $q \leqslant p$. The subset $\mathcal{D}$ is called *open* if it is open relative to the order topology of $\mathbb{P}$, i.e. if whenever $p \leqslant q$ with $q \in \mathcal{D}$ then $p \in \mathcal{D}$. A Banach space X is *generic* over the set $\mathbb{P}$ of finite approximations if $X = X_{\mathbb{F}}$ for some filter $\mathbb{F}$ of $\mathbb{P}$ that intersects *all* dense open subsets of $\mathbb{P}$. So, it follows that given a dense set $\mathcal{D}$, the collection $\mathbb{G} \cap \mathcal{D}$ is cofinal in $\mathbb{G}$, hence its direct limit coincide with the one obtained from $\mathbb{G}$, i.e. $\bigcup_{p\in\mathbb{G}\cap\mathbb{D}} X_p$ is dense in $X_{\mathbb{G}}$. We can say that "$X_{\mathbb{G}}$ has locally the property $\mathcal{D}$".

Observe that a generic filter can only be an imaginary which will not prevent us from using it in getting the corresponding generic Banach spaces $X_{\mathbb{G}}$ and studying their properties. In fact there is a whole new universe of sets, *the forcing extension*, that $\mathbb{G}$ generates using the sets from our *ground* universe V. In this paper we study the properties which the generic Banach space $X_{\mathbb{G}}$ has in the extended universe $V[\mathbb{G}]$ not only in the context of the poset $\mathbb{P}$ but also in the contexts of some of its natural variations. Much like in the context of classical extension of mathematical structures by adding imaginariness, one can study the extension $V[\mathbb{G}]$ as a collection of terms that involve ordinary sets and the single imaginary set $\mathbb{G}$. In fact in our context here all terms that we will ever use are so simple that all references to $V[\mathbb{G}]$ can be avoided via two basic facts which the reader not familiar with forcing could simply take as black-boxes having no disadvantage in following the rest of the paper. We pass now this discussion and give some examples of dense sets of the basic forcing $\mathbb{P}_{\text{basic}}$, and the implications this has on properties of the corresponding generic space.

Examples 3.10. (I) Given an ordinal $\gamma < \omega_1$, the set

$$\mathcal{D}_\gamma = \{p \in \mathbb{P}\colon\ \gamma \in D_p\}$$

is a dense-open subset of $\mathbb{P}_{\text{basic}}$: Fix a condition p, and suppose that $\gamma \notin D_p$. Let $q = (D_q, F_q, \sigma_q, F_q)$ be the basic condition defined by $D_q = D_p \cup \{\gamma\}$, $F_q = F_p \cup \{\pm u_\gamma\}$, $\sigma_q = \sigma_p \cup \{(0, u_\gamma)\}$, and for $s \in \sigma_q$, $h_s^{(q)} = h_s^{(p)}$ if $s \in \sigma_p$ and $h_{(0,\gamma)}^{(q)} = u_\gamma$. It is clear that $q \in \mathbb{P}_{\text{basic}}$, $q \leqslant p$ and that $q \in \mathcal{D}_\gamma$.

(II) Two basic conditions p and q are compatible if there is some basic condition r such that $r \leqslant p, q$. Otherwise, p and q are incompatible, and we write $p \perp q$. Let p be a basic condition. Let $p \in \mathbb{P}_{\text{basic}}$. The set

$$\mathcal{D}_p := \{q \in \mathbb{P}_{\text{b}}\colon\ q \perp p \text{ or } q \leqslant p\}$$

is dense.

(III) Given a limit ordinal $\alpha < \omega_1$ and $p \in (\mathbb{P}_{\text{b}})_\alpha$, the sets

$$\mathcal{D}_{p,\alpha,\infty} = \big\{q \in (\mathbb{P}_{\text{basic}})_\alpha\colon \text{ either } q \perp p \text{ or } q \leqslant p \text{ and } X_q \text{ is isometric to } \ell_\infty(D_q)\big\}$$

are dense in $(\mathbb{P}_{\text{basic}})_\alpha$: Fix a condition $q \in (\mathbb{P}_{\text{basic}})_\alpha$. Suppose that q is compatible with p and fix a basic condition $r \leqslant p, q$. Let $T : X_r \to \ell_\infty(F_r)$ be defined by $T(x) = (f(x))_{f\in F_r}$. This is an isometry. Let $Y = T(X_r)$. By using Remark 2.2(d) repeatedly we can find a sequence of $\mathbb{Q}$-vectors $(x_i, f_i)_{i<k}$ in $c_{00}(F_r)$ such that $\|x_i\|_\infty = \|f_i\|_{\ell_1} = 1$, $f_i(x_i) = 1$ for every $i < k$, $f_i(x_j) = f_i \restriction Y = f_0 \restriction Y = 0$ for every $j < i < k$, and such that $Y \oplus \langle x_i\rangle_{i<k} = \ell_\infty(F_r)$. We define the condition $r_0 = (D_0, F_0, H_0)$ as follows: $D_0 = D_r \cup \{\gamma_i\}_{i<k}$ where $D_r < \gamma_0 < \alpha$ and $\gamma_i := \gamma + i$ for each $i < k$. Let $U : c_{00}(D_r \cup \{\gamma_i\}_{i<k}) \to \ell_\infty(F_r)$ be the linear isomorphism onto

defined by $U \restriction X_r = T$ and $U(u_{\gamma_i}) = x_i$ for every $i < k$. Observe that for each $f \in F_r$ one has that $g_f := U^*(u_f^*)$ extends f, where u_f^* is the fth-functional in $\ell_\infty(F_r)$. Define

$$F_0 = \{\pm g_f\colon\ f \in F_r\} \cup \bigl\{\pm U^*(f_i)\colon\ i < k\bigr\}.$$

Define also $A_{r_0} = A_r$. Finally, for each $\gamma \in D_r$ let $h_\gamma^{(r_0)} = g_{h_\gamma^{(r)}}$, and for every $i < k$, let $h_{\gamma_i}^{(r_0)} = U^*(f_i)$. It is now routine to prove that $r_0 \leqslant r$. We check that $U : X_r \to \ell_\infty(F_r)$. Given $x \in X_{r_0}$ one has that

$$\|x\|_{r_0} = \max_{f \in F_r} \bigl|u_f^*\bigl(U(x)\bigr)\bigr| \vee \max_{i<k} \bigl|f_i\bigl(U(x)\bigr)\bigr| = \bigl\|U(x)\bigr\|_\infty. \tag{13}$$

(IV) Given a dense open subset $\mathcal{D}$ of $(\mathbb{P}_{\mathrm{basic}})_\alpha$, where α is a limit ordinal, the set

$$\mathcal{E}_{\mathcal{D}} = \{p \in \mathbb{P}_{\mathrm{basic}}\colon\ p \leqslant q \text{ for some } q \in \mathcal{D}\}$$

is dense in $\mathbb{P}_{\mathrm{basic}}$: Let p be any condition. Let I be the interval of ordinals $< \alpha$ with $\#(I) = \#(D_p \setminus \alpha)$ and such that $\min I = \max(D_p \cap \alpha) + 1$, and let $\theta : D_p \cup (D_p \cap \alpha) \cup I$ be the unique order-preserving mapping. Let $q \leqslant \theta(p)$ with $q \in \mathcal{D}$ ($\theta(p)$ is the θ-spread of p, see Definition 3.8). Now let J be an interval of ordinals with $\#(J) = \#(\{\gamma \in D_q\colon\ \gamma > \max I\})$, and such that $D_p < J$. Let $\pi : D_q \to (D_q \cap \min I) \cup (D_p \setminus \alpha) \cup J$ be the unique order-preserving bijection. Notice that by construction $\theta(\pi(q)) = q$ and $\pi(\theta(p)) = p$ (because $\pi \restriction D_p \cap \alpha = \mathrm{Id}$, and $\pi(I) = D_p \setminus \alpha$). So, $\pi(q) \leqslant \pi(\theta(p)) = p$. The conditions q and $\pi(q)$ are compatible (see Proposition 3.27), so there is some $r \leqslant q, \pi(q)$, and hence $r \in \mathcal{E}$ and $r \leqslant \pi(q) \leqslant \pi(\theta(p)) = p$.

(V) Given a dense open subset $\mathcal{D}$ of $(\mathbb{P}_{\mathrm{basic}})_H$, the set

$$\mathcal{E}_{\mathcal{D}} = \{p \in \mathbb{P}_{\mathrm{basic}}\colon\ p_H \in \mathcal{D}\}$$

is dense in $\mathbb{P}_{\mathrm{basic}}$: Fix a condition $p \in \mathbb{P}_{\mathrm{basic}}$. Since $\mathcal{D}$ is dense in $(\mathbb{P}_{\mathrm{basic}})_H$, there is some $q \in \mathcal{D}$ with $q \leqslant p_H$. Then $r = (D_q, F_p \cup \pm H_q, A_q, H_q)$ is a basic condition such that $r_H = q$ and $r \leqslant q$: $D_p \subseteq D_q = D_r$, $H_p \subseteq H_q \restriction D_p \subseteq \mathrm{conv}_{\mathbb{Q}}(\pm H_p)$ and $F_p \subseteq F_r \restriction D_p = H_q \restriction D_p \cup F_p \subseteq \mathrm{conv}_{\mathbb{Q}}(\pm H_p \restriction D_p) \cup F_p \subseteq \mathrm{conv}_{\mathbb{Q}}(F_p)$.

The following result tells that the separable subspaces $X_{\mathbb{G}}^{(\alpha)}$ of $X_{\mathbb{G}}$ are also generic spaces in a very natural way.

Proposition 3.11. *Suppose that $\mathbb{G}$ be a generic filter of $\mathbb{P}_{\mathrm{basic}}$. Then*

(a) $\mathbb{G}_H := \{p_H\colon\ p \in \mathbb{G}\}$ *is a generic filter for $\mathbb{P}_{\mathrm{basic}}$. In particular, for every $x \in c_{00}(D_{\mathbb{F}})$ one has that $\|x\|_{\mathbb{G},H} = \|x\|_{\mathbb{G}_H}$. Hence*

$$X_{\mathbb{G},H} = X_{\mathbb{G}_H}.$$

(b) *For every limit ordinal $\alpha < \omega_1$ the subset $\mathbb{G}_\alpha := \mathbb{G} \cap (\mathbb{P}_{\mathrm{basic}})_\alpha$ of $\mathbb{P}_\alpha$ is a generic filter of $(\mathbb{P}_{\mathrm{basic}})_\alpha$. In particular, for every $x \in c_{00}(\alpha)$ one has that $\|x\|_{\mathbb{G}_\alpha} = \|x\|_{\mathbb{G}}$. Hence*

$$X_{\mathbb{G}}^{(\alpha)} = X_{\mathbb{G}_\alpha}.$$

Proof. We have to prove that $\mathbb{G}_\alpha$ is a generic filter for the forcing $(\mathbb{P}_{\text{basic}})_\alpha$. If $p \in \mathbb{G}_\alpha$ and $p \leqslant q$ then clearly $q \in \mathbb{G} \cap (\mathbb{P}_{\text{basic}})_\alpha = \mathbb{G}_\alpha$. Suppose that $p, q \in \mathbb{G}_\alpha$. Let $r \leqslant p, q$, $r \in \mathbb{G}$. This condition does not need to be in $(\mathbb{P}_{\text{basic}})_\alpha$, but the condition $r \restriction \alpha$ is in $(\mathbb{P}_{\text{basic}})_\alpha$, $r \in \mathbb{G}$ (because $r \leqslant r \restriction \alpha$ and $\mathbb{G}$ is a filter) and $r \restriction \alpha \leqslant p \restriction \alpha = p, q \restriction \alpha = q$. Finally, we prove that $\mathbb{G}_\alpha$ is a generic filter for $(\mathbb{P}_{\text{basic}})_\alpha$: Let $\mathcal{D}$ be a dense-open set of $(\mathbb{P}_{\text{basic}})_\alpha$. We know that $\mathcal{E}_{\mathcal{D}}$ is a dense set of $\mathbb{P}_{\text{basic}}$ (see Example 3.10(IV)). Hence there is some $p \in \mathbb{G}$ such that $p \leqslant q$ for some $q \in \mathcal{D}$. Since $\mathbb{G}$ is a filter, it follows that $q \in \mathbb{G} \cap \mathcal{D} \subseteq \mathbb{G}_\alpha$, as desired. □

The next result is also proved by density arguments.

Proposition 3.12. *Let $\mathbb{G}$ be a generic filter for the basic forcing notion $\mathbb{P}_{\text{basic}}$. Then*:

(I) $D_{\mathbb{G}} = \omega_1$.
(II) *For every limit $\alpha < \omega_1$ the subspace $X_{\mathbb{G}}^{(\alpha)}$ of $X_{\mathbb{G}}$ is the separable Gurarij space.*

Before we start the proof, we need to give a characterization of Gurarij spaces. Having in mind the already mentioned characterization of separable Lindenstrauss spaces, we give a corresponding characterization of separable Gurarij spaces.

Proposition 3.13. *A separable Banach space X is the Gurarij space if and only if there is a $\subseteq$-directed family $\mathcal{F}$ of finite-dimensional subspaces of X isometric to corresponding $\ell_\infty^{\dim X}$ such that the union $\bigcup \mathcal{F} = \bigcup_{F \in \mathcal{F}} F$ is dense in X and such that*:

(+) *Whenever that $F \in \mathcal{F}$ and $T : F \to \ell_\infty^n$ is a $\mathbb{Q}$-isometry then there is $F \subseteq G \in \mathcal{F}$ and an onto $\mathbb{Q}$-isometry $U : G \to \ell_\infty^n$ extending T.*

If such a family exists for an arbitrary, not necessarily separable, Banach space X then X is a Gurarij space.

Proof. Suppose first that such family $\mathcal{F}$ exists. The proof of that X is Gurarij is standard: let $G \subseteq H$ be two f.d. normed spaces, and let $T : G \to X$ be an isometry. First of all, since $\bigcup \mathcal{F}$ is dense in X without loss of generality we may assume that there is some $F \in \mathcal{F}$ such that $T : G \to F$. Let Y be the quotient space of $F \oplus_1 H$, the cartesian product of F and H equipped with the norm $\|(f, h)\| = \|f\|_F \oplus \|h\|_H$, modulo its subspace $N = \{(T(g), -g):\ g \in G\}$. Then $f \in F \mapsto i_0(f) := (f, 0) + N \in Y$ and $h \in H \mapsto i_1(g) := (0, g) + N \in Y$ are both isometries. By a simple approximation argument, we may further assume that there is an isometry $U : Y \to \ell_\infty^n$ for some n. Let $V : F \to \ell_\infty^n$ be an appropriate approximation of $U \circ i_0$ which is a $\mathbb{Q}$-isometry. By (+) there is some $F \subseteq F_1 \in \mathcal{F}$ and a $\mathbb{Q}$-isometry onto $V_1 : F_1 \to \ell_\infty^n$ extending V. Then $V_1^{-1} \circ U \circ i_1 : H \to F_1$ is an isometry, and its sufficiently fine approximation $T_1 : H \to F_1$ is the desired extension of T.

The direct implications is consequence of the uniqueness of the separable Gurarij spaces, and the fact that, as we are going to see, the generic space $X_{\mathbb{G}}^{(\omega)}$ has the desired family, and consequently is the Gurarij space. □

Proof of Proposition 3.13. (I): Fix $\gamma \in \omega_1$. Since $\mathbb{G}$ is generic, it meets the dense set $\mathcal{D}_\gamma$. So, there is some $p \in \mathbb{G}$ such that $\gamma \in D_p \subseteq D_{\mathbb{G}}$, as desired.

(II): We use that $X_{\mathbb{G}}^{(\alpha)} = X_{\mathbb{G}_\alpha}$. Let

$$\mathcal{F} := \{F:\ F \subseteq X_p \text{ for some } p \in \mathbb{G}_\alpha \text{ and } F \text{ is isometric to } \ell_\infty^{\dim F}\}.$$

We have proved in Example 3.10(III) that for every $p \in \mathbb{P}_{\mathrm{b}}$ the set of basic conditions q which are either incompatible with p or $q \leqslant p$ and with X_q isometric to corresponding ℓ_∞^n is dense in $\mathbb{P}_{\mathrm{basic}}$. This fact readily implies that $\mathcal{F}$ is directed and that $\bigcup \mathcal{F}$ is dense in $X_{\mathbb{G}}^{(\alpha)}$. It rest to show that $\mathcal{F}$ has the property (+) above. To prove this, we define the following sets. Let $p \in \mathbb{P}_\alpha$ and suppose that $T : F \to \ell_\infty^n$ is a $\mathbb{Q}$-isometry, with $F \subseteq X_p$. Define then $\mathcal{D}_{p,T}$ as the set of basic conditions $q \in \mathbb{P}_\alpha$ such that either $q \perp p$ or $q \leqslant p$ and there is $F \subseteq G \subseteq X_q$ isometric to ℓ_∞^n and an isometry onto $U : G \to \ell_\infty^n$ extending T.

Claim 3.13.1. *$\mathcal{D}_{p,T}$ is dense in $\mathbb{P}_\alpha$.*

We rapidly sketch the proof of this fact: We fix $r \leqslant p$. Let $(x_i, f_i)_{i<k}$ be a sequence in $\ell_\infty^n \times \ell_1^n$ such that $\|x_i\| = \|f_i\| = 1$, $f_i(x_i) = 1$, and $f_j(x_i) = f_j \upharpoonright T(F) = F_0 \upharpoonright T(F) = 0$ for every $i < j < k$. Let $D_r < I < \alpha$ be an interval of cardinality k, $I = \{\gamma_i\}_{i<k}$ an enumeration of it. For each $f \in F_r$, let $g_f \in \ell_1^n$ of norm $\leqslant 1$ be such that $T^*(g_f) = f \upharpoonright F$. And for each $i < n$, let $h_i \in F_r$ be such that $h_i \upharpoonright F = T^*(u_i^*)$. Define $q = (D_r \cup I, F_q, A_q, H_r)$, where $F_q = \{f \vee \bigvee_{i<k} g_f(x_i)u_{\gamma_i}:\ f \in F_r\} \cup \{\pm \sum_{i<k} f_j(x_i)u_{\gamma_i}:\ j < k\} \cup \{\pm h_i \vee \bigvee_{j<k} (x_j)_i u_{\gamma_i}:\ i < n\}$, and $h_\gamma^{(q)} = h_\gamma^{(r)} \vee \bigvee_{i<k} g_{h_\gamma^{(r)}}(x_i)u_{\gamma_i}$ for $\gamma \in D_r$, and $h_{\gamma_i}^{(q)} := \sum_{j<k} f_i(x_i)u_{\gamma_i}$ for every $i < k$. Then if we set $G := F + \langle u_{\gamma_i} \rangle_{i<k}$, the linear extension $U : G \to \ell_\infty^n$, $U \upharpoonright F = T$ and $U(u_{\gamma_i}) = x_i$ is an isometry onto. This implies that $q \in \mathcal{D}_{p,T}$.

It is not difficult to see that the claim gives property (+) for the family $\mathcal{F}$. $\square$

3.2. *Extensions of conditions*

We define now a class of forcing notions having the properties of $\mathbb{P}_{\mathrm{basic}}$ exposed in Proposition 3.11 and Proposition 3.12. This is done by extracting some natural properties of the basic forcing notion which allow us to prove that the sets introduced in Examples 3.10 are dense.

Definition 3.14. Let $\mathbb{P}$ be a given forcing notion. We say that a forcing notion $\mathbb{P}$ is *hereditary* if for every $p \in \mathbb{P}$ and every $\alpha < \omega_1$ one has that $p \upharpoonright \alpha$ is in $\mathbb{P}$. We say that $\mathbb{P}$ is *spreading* if for every condition $p \in \mathbb{P}$ and every $\theta : D_p \to \omega_1$ which is order-preserving and successor preserving (i.e. $\theta(\gamma + 1) = \theta(\gamma) + 1$ for every $\gamma, \gamma + 1 \in D_p$), the corresponding θ-spread $\theta(p)$ of p (see Definition 3.8) is also in $\mathbb{P}$.

It is clear that the basic forcing $\mathbb{P}_{\mathrm{basic}}$ is both hereditary and spreading.

Definition 3.15. Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ has the *extension property* (EP) when

(1) $\mathbb{P}$ is hereditary and spreading.
(2) Let $p = (D_p, F_p, A_p, H_p)$ be in $\mathbb{P}$. Suppose that q is a basic condition such that $q \leqslant p$, $A_q = A_p$ and $h_\gamma^{(q)} = u_\gamma$ for every $\gamma \in D_q \setminus D_p$. Then the condition q is in $\mathbb{P}$.

It is clear that the basic forcing $\mathbb{P}_{\text{basic}}$ has the (EP). It is easy to see that if $\mathbb{P}$ has the (EP), then so does $\mathbb{P}_H$. We list some useful properties of forcing notions with the (EP).

Proposition 3.16. *Suppose that $\mathbb{P}$ has the* (EP). *Let $\mathbb{G}$ be a generic filter for it, and let $\alpha < \omega_1$ be a limit ordinal. Then*

(1) $D_{\mathbb{G}} = \omega_1$.
(2) $\mathbb{G}_\alpha = \mathbb{G} \cap \mathbb{P}_\alpha$ *is a generic filter for $\mathbb{P}_\alpha$, and consequently, $X_{\mathbb{G}_\alpha} = X_{\mathbb{G}}^{(\alpha)}$.*
(3) *The spaces $X_{\mathbb{G}}^{(\alpha)}$ and $X_{\mathbb{G},H}^{(\alpha)}$ are both infinite-dimensional Lindenstrauss spaces. Hence, $X_{\mathbb{G}}$ and $X_{\mathbb{G},H}$ are also Lindenstrauss spaces.*

Proof. The proofs presented in Proposition 3.11 and Proposition 3.12 for the basic forcing also work for $\mathbb{P}$ because of the hypothesis in $\mathbb{P}$. □

The following result explains the role of the previous classes of forcing notions and the differences between the generic spaces $X_{\mathbb{G}}$ and $X_{\mathbb{G},H}$. Note that since $X_{\mathbb{G}}^{(\alpha)}$ is Gurarij, its dual space is isometric to $M[0,1]$. On the other hand, being $X_{\mathbb{G},H}^{(\alpha)}$ an $\mathcal{L}_\infty$-space, its dual space has to be either isomorphic to $M[0,1]$ or to ℓ_1. We are going to prove that the second case holds in a strong sense.

Theorem 3.17. *Suppose that $\mathbb{P}$ has the* (EP), *let $\mathbb{G}$ be a generic filter for it. Then for every limit ordinal $\alpha < \omega_1$ the sequence $(h_\gamma^{(\mathbb{G})} \upharpoonright \alpha)_{\gamma<\alpha}$ is a Schauder basis of $(X_{\mathbb{G},H}^{(\alpha)})^*$ which is* 1*-equivalent to the unit basis of $\ell_1(\alpha)$. Consequently*:

(1) *The sequence $(h_\gamma^{(\mathbb{G})})_{\alpha<\omega_1}$ is a basis of $(X_{\mathbb{G},H})^*$* 1*-equivalent to the unit basis of $\ell_1(\omega_1)$.*
(2) *The space $X_{G,H}$ is Asplund and c_0-saturated.*

Proof. Fix a limit ordinal $\alpha < \omega_1$. We first prove that $(h_\gamma^{(\mathbb{G})} \upharpoonright \alpha)_{\alpha<\omega_1}$ is 1-equivalent to the unit basis of $\ell_1(\alpha)$, i.e. for every $s \subseteq \alpha$ and every sequence of scalars $(a_\gamma)_{\gamma\in s}$ one has that

$$\left\| \sum_{\gamma\in s} a_\gamma h_\gamma^{(\mathbb{G})} \upharpoonright \alpha \right\|_{(X_{\mathbb{G},H}^{(\alpha)})^*} = \sum_{\gamma\in s} |a_\gamma|.$$

Given $p \in \mathbb{P}_\alpha$, let

$$\mathcal{D}_p := \left\{ q \in \mathcal{P}_\alpha \colon\ q \perp p \text{ or } q \leqslant p \text{ and } \left(h_\gamma^{(q)}\right)_{\gamma\in D_p} \text{ is 1-equivalent to the unit basis of } \ell_1(D_p) \right\}.$$

Claim 3.17.1. *$\mathcal{D}_p$ is dense in $\mathbb{P}_\alpha$.*

Proof. Fix $q \in \mathbb{P}_\alpha$ and suppose that q is compatible with p. Let $r \leqslant q, p$. Let $\gamma_0 = \max D_r$, and let $\gamma_0 < I < \alpha$ be an interval of ordinals of cardinality $2^{\#(D_r)}$, $I = \{\gamma_\varepsilon\}_{\varepsilon\in\{-1,1\}^{D_r}}$ an enumeration of it. We define $r_0 = (D_r \cup I, F_0, H_0)$, where $F_0 = F_r \cup \{\pm h_\gamma\}_{\gamma\in D_r} \cup \{\pm h_\gamma\}_{\gamma\in I}$, and

$$h_\gamma := \begin{cases} u_\gamma & \text{if } \gamma \in I, \\ h_\gamma^{(r)} \vee \bigvee_{\varepsilon\in\{-1,1\}^{D_r}} \varepsilon(\gamma) \cdot u_{\gamma_\varepsilon} & \text{if } \gamma \in D_r. \end{cases}$$

Then, because of (EP), $r_0 \in \mathbb{P}_\alpha$. It is clear that $r_0 \leqslant r$ and it is easy to see that $(h_\gamma)_{\gamma \in D_r} \subseteq X^*_{r_0,H}$ is 1-equivalent to the unit basis of $\ell_1(D_r)$. $\square$

Now let $s \subseteq \alpha$ be finite and let $(a_\gamma)_{\gamma \in s}$. Let $p \in \mathbb{G}$ be such that $s \subseteq D_p$ and let $q \in \mathbb{G} \cap D_p$. Then,

$$\sum_{\gamma \in s} |a_\gamma| \geqslant \Bigl\| \sum_{\gamma \in s} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \alpha \Bigr\|_{(X_{\mathbb{G},H})^*} \geqslant \Bigl\| \sum_{\gamma \in s} a_\gamma h_\gamma^{(q)} \restriction \alpha \Bigr\|_{(X_{q,H})^*} = \sum_{\gamma \in s} |a_\gamma|. \tag{14}$$

The first part of the proof is finished once we prove the following:

Claim 3.17.2. *For every $f \in (X^{(\alpha)}_{\mathbb{G},H})^*$ there is a sequence $(a_\gamma)_{\gamma<\alpha} \in \ell_1(\alpha)$ such that*

$$f = \sum_{\gamma<\alpha} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \alpha. \tag{15}$$

Proof. This is done by induction on α limit. We use the consequence of (EP) that given $p \in \mathbb{P}_{\gamma+\omega}$ the set

$$\mathcal{D}_p := \bigl\{ p \in \mathbb{P}_{\gamma+\omega}\colon\ D_p \cap [\gamma, \gamma+\omega[\text{ is an initial interval of } [\gamma, \gamma+\omega[\bigr\}$$

is dense in $\mathbb{P}_{\gamma+\omega}$. Its proof is not difficult and so we leave the details to the reader. Suppose that $\alpha = \beta + \omega$ with β limit, including $\beta = 0$. By inductive hypothesis we know that

$$f \restriction \beta = \sum_{\gamma<\beta} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \beta. \tag{16}$$

Set

$$g = \frac{1}{2}\Bigl(f - \sum_{\gamma<\beta} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \alpha \Bigr).$$

Since $(h_\gamma^{(\mathbb{G})\restriction\beta})_{\gamma<\beta}$ and $(h_\gamma^{(\mathbb{G})\restriction\alpha})_{\gamma<\alpha}$ are 1-equivalent to the corresponding unit bases of $\ell_1(\beta)$ and $\ell_1(\alpha)$, it follows from (16) that

$$\begin{aligned}\Bigl\| \sum_{\gamma<\beta} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \alpha \Bigr\|_{(X^{(\alpha)}_{\mathbb{G},H})^*} &= \sum_{\gamma<\beta} |a_\gamma| = \Bigl\| \sum_{\gamma<\beta} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \beta \Bigr\|_{(X^{(\beta)}_{\mathbb{G},H})^*} \\ &= \| f \restriction \beta \|_{(X^{(\beta)}_{\mathbb{G},H})^*} \leqslant \|f\|_{(X^{(\alpha)}_{\mathbb{G},H})^*} \leqslant 1,\end{aligned}$$

so g is in the dual unit ball of $X_{\mathbb{G},H}$. Note that $g \restriction \beta = 0$. Let $(p_n)_n$ be a sequence of conditions in $\mathbb{P}_\alpha$ such that $p_{n+1} \leqslant p_n$ and such that $D_{p_n} \cap [\beta, \beta+\omega[$ is an initial interval of $[\beta, \beta+\omega[$. Fix n. Since $g \restriction D_{p_n}$ is in the dual unit ball of $X_{p_n,H}$, Proposition 3.4(d) gives that there is a unique representation

$$g \restriction D_{p_n} = \sum_{\gamma \in D_{p_n}} a_\gamma^{(n)} h_\gamma^{(p_n)}.$$

Since $g \restriction \beta = 0$, it follows that $a_\gamma^{(n)} = 0$ for every $\gamma \in D_{p_n} \cap \beta$. Now for $m < n$ one has that

$$g \restriction D_{p_m} = g \restriction D_{p_n} \restriction D_{p_m} = \sum_{\gamma \in D_{p_n}} a_\gamma^{(n)} h_\gamma^{(p_n)} \restriction D_{p_m} = \sum_{\gamma \in D_{p_m}} a_\gamma^{(n)} h_\gamma^{(p_m)}, \tag{17}$$

the last equality because for $\gamma \in D_{p_n} \setminus D_{p_m}$, if $\gamma < \beta$, then $a_\gamma^{(n)} = 0$, and if $\gamma \in [\beta, \beta + \omega[$, then $D_{p_m} < \gamma$ and so $h_\gamma^{(p_n)} \restriction D_{p_m} = 0$. Hence for $m < n$ and $\gamma \in D_{p_m}$ one has that

$$a_\gamma^{(n)} = a_\gamma^{(m)}. \tag{18}$$

Let $(a_\gamma)_{\beta \leqslant \gamma < \alpha}$ be the sequence defined by $a_\gamma = a_\gamma^{(n)}$ for some n such that $\gamma \in D_{p_n}$. Then $(a_\gamma)_\gamma$ is summable, and

$$g = \sum_{\beta \leqslant \gamma < \alpha} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \alpha.$$

So,

$$f = \sum_{\gamma < \beta} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \alpha + 2 \sum_{\beta \leqslant \gamma < \alpha} a_\gamma h_\gamma^{(\mathbb{G})}.$$

Finally, if α is a limit of limits, $\alpha = \sup_n \alpha_n$, with $\alpha_n < \alpha_{n+1}$, then for each n one has that

$$f \restriction \alpha_n = \sum_{\gamma < \alpha_n} a_\gamma^{(n)} h_\gamma^{(\mathbb{G})} \restriction \alpha_n$$

with $(a_\gamma^{(n)})_{\gamma < \alpha_n} \in \ell_1(\alpha_n)$. It follows that $a_\gamma^{(m)} = a_\gamma^{(n)}$ for every $\gamma < \alpha_m < \alpha_n$, hence if we set $a_\gamma = a_\gamma^{(n)}$ with n such that $\gamma < \alpha_n$, then

$$f = \sum_{\gamma < \alpha} a_\gamma h_\gamma^{(\mathbb{G})},$$

as desired. This ends the first part of the proof. We prove now (1): It is clear that $(h_\gamma^{(\mathbb{G})})_{\gamma < \omega_1}$ is the ℓ_1-basis. To prove that it is a basis of the dual space, we fix $f \in X_{\mathbb{G},H}$. Then by the previous result, for every limit $\alpha < \omega_1$ there is a unique sequence $(a_\gamma^{(\alpha)})_{\gamma < \alpha} \in \ell_1(\alpha)$ such that

$$f \restriction \alpha = \sum_{\gamma < \alpha} a_\gamma h_\gamma^{(\mathbb{G})} \restriction \alpha.$$

Hence, $a_\gamma^{(\alpha)} = a_\gamma^{(\beta)}$ for every $\gamma < \alpha < \beta$. So, we can define $(a_\gamma)_{\gamma < \omega_1}$ by $a_\gamma := a_\gamma^{(\alpha)}$ where $\gamma < \alpha$ is limit. Then $(a_\gamma)_{\gamma < \omega_1}$ is summable, and $f = \sum_{\gamma < \omega} a_\gamma h_\gamma^{(\mathbb{G})}$.

(2): The two required properties are separably determined, so it suffices to check them on each $X_{\mathbb{G},H}^{(\alpha)}$ with $\alpha < \omega_1$ limit. For such α, the dual space of $X_{\mathbb{G},H}^{(\alpha)}$ is isometric to $\ell_1(\alpha)$, so $X_{\mathbb{G},H}^{(\alpha)}$ is Asplund. Because the set of extremal points of the unit ball of $\ell_1(\alpha)$ is precisely the countable set $\{\pm u_\gamma\}_{\gamma < \alpha}$, it follows by a well-known result of V.P. Fonf [7] that $X_{\mathbb{G},H}^{(\alpha)}$ is c_0-saturated. $\square$

So, for partial orderings with the (EP), the corresponding generic spaces $X_{\mathbb{G}}$ and $X_{\mathbb{G},H}$ are different. The next result gives the exact relation between the two generic spaces.

Let $\mathbb{P}$ be a forcing notion with (EP). Let $\mathbb{G}$ be a generic filter for $\mathbb{P}$. Let

$$i : \big(c_{00}(\omega_1), \|\cdot\|_{\mathbb{G}}\big) \to \big(c_{00}(\omega_1), \|\cdot\|_{\mathbb{G},H}\big)$$

be the identity mapping. This is a bounded operator, indeed of norm 1. Let

$$\pi_{\mathbb{G}} : X_{\mathbb{G}} \to X_{\mathbb{G},H}$$

be its extension to the corresponding completions. Let also

$$N_{\mathbb{G}} := \operatorname{Ker}(\pi_{\mathbb{G}}).$$

Theorem 3.18. *Suppose that $\mathbb{P}$ has the* (EP)*, and let $\alpha < \omega_1$ be a limit ordinal. Then*

(I) $N_{\mathbb{G}} \subseteq X_{\mathbb{G}}^{(\omega)}$ *is the separable Gurarij.*
(II) $\pi_{\mathbb{G}} : X_{\mathbb{G}}^{(\alpha)} \to X_{\mathbb{G},H}^{(\alpha)}$ *is a quotient map, and*

$$\begin{aligned} \widetilde{\pi}_{\mathbb{G}} : X_{\mathbb{G}}^{(\alpha)}/N_{\mathbb{G}} &\to X_{\mathbb{G},H}^{(\alpha)} \\ x + N_{\mathbb{G}} &\mapsto \pi_{\mathbb{G}}(x) \end{aligned}$$

is an isometry onto.

Proof. (I): We prove that $N_{\mathbb{G}} \subseteq X_{\mathbb{G}}^{(\omega)}$, so separable. We introduce two sets of conditions. Let $p \in \mathbb{P}$, $\varepsilon > 0$, $\vec{x} = (x_i)_{i<k+l}$ be a normalized $\mathbb{Q}$-basis, and let $\vec{v} = (v_i)_{i<k}$ be a $\mathbb{Q}$-sequence of X_p such that

$$(v_i)_{i<k} \sim_1 (x_i)_{i<k} \quad \text{and} \quad \Big\|\sum_{i<k+l} a_i x_i\Big\| \geqslant \Big\|\sum_{i<k} a_i v_i\Big\|_{p,H} \quad \text{for all } (a_i)_{i<k+l}. \tag{19}$$

We define $\mathcal{D}_{p,\vec{v},\vec{x},\varepsilon}$ as the set of all conditions $q \in \mathbb{P}$ such that either q is incompatible with p or else $D_q \cap \omega$ is an initial interval of ω, and there is a sequence $(\bar{w}_i)_{i<l}$ of $\mathbb{Q}$-points of $X_q \cap c_{00}(\omega)$ such that

(i) $(v_0, \dots, v_{k-1}, \bar{w}_0, \dots, \bar{w}_{l-1}) \sim_1 (x_0, \dots, x_{k+l-1})$.
(ii) $\|\bar{w}_i\|_{H_q} \leqslant \varepsilon$ for every $i < l$.

In addition, given $(w_i)_{i<l}$ a $\mathbb{Q}$-sequence in X_p such that

$$(v_i)_{i<k+l} \sim_1 (v_0, \dots, v_{k-1}, w_0, \dots, w_{l-1}), \tag{20}$$

we define $\mathcal{E}_{p,\vec{v},\vec{w},\vec{x},\varepsilon}$ as the set of all conditions $q \in \mathbb{P}$ such that either q is incompatible with p or else $D_q \cap \omega$ is an initial interval of ω, and there is a sequence $(\bar{w}_i)_{i<l}$ of $\mathbb{Q}$-points of $X_q \cap c_{00}(\omega)$ such that (i) and (ii) above hold and in addition

(iii) $\|\bar{w}_i - w_i\|_q \leqslant \max\{\|w_i\|_{p,H}, \varepsilon\}$ for every $i < l$.

Observe that if q belongs to one of the above dense sets then it follows from the fact that $D_q \cap \omega$ is an initial interval of ω that if $\mathbb{G}$ is a generic filter with $q \in \mathbb{G}$, then

(i′) $\|\bar{w}_i\|_{H_{\mathbb{G}}} \leqslant \varepsilon$ for every $i < k$.

Claim 3.18.1. *$\mathcal{D}_{p,\vec{v},\vec{x},\varepsilon}$ and $\mathcal{E}_{p,\vec{v},\vec{w},\vec{x},\varepsilon}$ are both dense in $\mathbb{P}$.*

Proof. Fix all initial data $p, \vec{v}, \vec{w}, \vec{x}$ and $\varepsilon > 0$ fulfilling the hypothesis, and fix $r \in \mathbb{P}$. Without loss of generality we may assume that $r \leqslant p$. Let M be a $\mathbb{Q}$-f.d. space with $\vec{x} = (x_i)_{i<k+l}$ in M. Let $I \subseteq \omega \setminus D_r$ be a finite set of cardinality $\geqslant \sum_{i<l} \|x_i^*\|/\varepsilon$ and such that $(D_r \cap \omega) \cup I$ is an initial interval of ω. Let $(I_i)_{i<l}$ be pairwise-disjoint subsets of I with $\#(I_i) \geqslant \|x_i^*\|/\varepsilon$ for every $i < l$. For each $f \in F_p$ let $g_f \in B_{M^*}$ be a $\mathbb{Q}$-extension of the functional $\bar{f}$ defined linearly on $\langle x_i \rangle_{i<k+l}$ by $\bar{f}(x_i) = f(v_i)$ for every $i < k + l$. And for each $g \in \mathrm{Ext}(B_{M^*})$, let $f_g \in B_{X_p^*}$ be extending the functional $\bar{f}$ of $\langle v_0, \dots, v_{k-1}, w_0, \dots, w_{l-1} \rangle$ defined linearly by $\bar{f}(v_i) = g(x_i)$ for $i < k$ and $\bar{f}(w_i) = g(x_{k+i})$ for $i < l$. Define $D_q = (D_q, F_q, A_q, H_q)$ where

(a) $D_q = D_r \cup I$.
(b) F_q is the minimal symmetric set containing
  (b.1) $f + \sum_{i<l} g_f(x_{k+i})\chi_{I_i}$ for every $f \in F_r$,
  (b.2) $h_\gamma^{(r)}$ for every $\gamma \in A_r$,
  (b.3) $f_g + \sum_{i<l} g(x_{k+i})\chi_{I_i}$ for every $g \in \mathrm{Ext}(B_{M^*})$, and
  (b.4) u_γ for every $\gamma \in I$.
(c) $A_q = A_r$.
(d) We define $h_\gamma^{(q)} = h_\gamma^{(r)}$ for $\gamma \in D_r$ and $h_\gamma^{(q)} = u_\gamma$ for $\gamma \in I$.

Then $q \leqslant r$ is in $\mathbb{P}$ because of the (EEP) of $\mathbb{P}$. We check that $q \in \mathcal{D}_{p,\vec{v},\vec{x},\varepsilon} \cap \mathcal{E}_{p,\vec{v},\vec{w},\vec{x},\varepsilon}$: Let $\bar{w}_i = (1/\#I_i)\chi_{I_i}$ for every $i < l$. Then (i), (ii) and (iii) hold for $(\bar{w}_i)_{i<l}$. We leave the details to the reader. $\square$

Using this claim we prove that $N_{\mathbb{G}} \subseteq X_{\mathbb{G}}^{(\omega)}$. So, fix $x \in N_{\mathbb{G}}$ with $\|x\|_{\mathbb{G}} = 1$, and $\varepsilon > 0$. Let $w \in c_{00}(\omega_1, \mathbb{Q})$ with $\|w\|_{\mathbb{G}} = 1$ and $\|w - x\|_{\mathbb{G}} < \varepsilon/2$. Observe that this last fact implies that

$$\text{for every } \gamma < \omega_1 \text{ one has that } \left|h_\gamma^{(\mathbb{G})}(v)\right| < \frac{\varepsilon}{2}. \tag{21}$$

Now find $p \in \mathbb{G}$ with $w \in X_p$. Since $\mathcal{D}_{p,(w),\emptyset,\varepsilon/2}$ is dense in $\mathbb{P}$ and $\mathbb{G}$ is a generic filter, there is $q \in \mathcal{D}_{p,\emptyset,(w),\emptyset,\varepsilon/2} \cap \mathbb{G}$ with $q \leqslant p$. Hence there is some $\mathbb{Q}$-point $\bar{w}$ in $X_q \cap c_{00}(\omega)$ such that

$$\|\bar{w} - w\|_{\mathbb{G}} = \|\bar{w} - w\|_q \leqslant \frac{\varepsilon}{2}, \tag{22}$$

the last inequality because of (21) and (iii).

We now prove that $N_{\mathbb{G}}$ is a Gurarij space. It is easy to see that this is equivalent to prove the following:

($*$) Let $k \in \mathbb{N}$, $(x_i)_{i<k+l}$ be a basic sequence in a Banach space, $\|x_i\| \leqslant 1$, and $(v_i)_{i<k}$ be points in $N_{\mathbb{G}}$ such that $(x_i)_{i<k}$ is 1-equivalent to $(v_i)_{i<k}$. Then there are points $(z_i)_{i<l}$ in $N_{\mathbb{G}}$ such that $(x_0, \dots, x_{k-l-1})$ is $1 + \varepsilon$-equivalent to $(v_0, \dots, v_{k-1}, z_0, \dots, z_{l-1})$.

Now using this characterization and a simple approximation argument, we reduce our task to proving the following:

$(**)$ Given $\varepsilon > 0$, a $\mathbb{Q}$-basis $(x_i)_{i<k+l}$, $\|x_i\| \leqslant 1$ for all $i < k+l$, then there is $\delta = \delta(\varepsilon) > 0$ such that whenever $(v_i)_{i<k}$ is a sequence in $c_{00}(\omega_1, \mathbb{Q})$ with $\|v_i\|_{\mathbb{G},H} \leqslant \delta$ and such that $(x_i)_{i<k}$ is 1-equivalent to $(v_i)_{i<k}$, then there are points $(\bar{w}_i)_{i<l}$ in $N_{\mathbb{G}}$ such that $(x_0, \dots, x_{k+l-1})$ is 1-equivalent to $(v_0, \dots, v_{k-1}, \bar{w}_0, \dots, \bar{w}_{l-1})$.

So, we fix the initial data in $(**)$, $\varepsilon > 0$ and a $\mathbb{Q}$-basis $(x_i)_{i<k+l}$, $\|x_i\| \leqslant 1$ for all $i < k+l$. Let

$$\delta = \min\left\{\varepsilon, \frac{1}{k \max_{i<k} \|x_i^*\|}\right\}, \tag{23}$$

where x_i^* is the functional such that $x_i^*(x_j) = \delta_{i,j}$ for every $i, j < k$. Let now $(v_i)_{i<k}$ be a sequence in $c_{00}(\omega_1, \mathbb{Q})$ with $\|v_i\|_{\mathbb{G},H} \leqslant \delta$ and such that $(x_i)_{i<k} \sim_1 (v_i)_{i<k}$, and let $p \in \mathbb{G}$ be such that $v_i \in X_p$ for all $i < k$. We check that (19) holds, and for this we fix a sequence of scalars $(a_i)_{i<k+l}$. Then

$$\left\| \sum_{i<k+l} a_i x_i \right\| \geqslant \max_{i<k} \frac{|a_i|}{\|x_i^*\|} \geqslant \frac{\max_{i<k} |a_i|}{\max_{i<k} \|x_i^*\|} \geqslant \delta k \max_{i<k} |a_i| \geqslant \left\| \sum_{i<k} a_i v_i \right\|_{p,H}.$$

Let $q_0 \in \mathcal{D}_{p,\vec{v},\vec{x},\delta}$, and let $(w_i^{(0)})_{i<l}$ be witnessing that q_0 is in that dense set. It follows that

(a) $(v_0, \dots, v_{k-1}, w_0^{(0)}, \dots, w_{l-1}^{(0)})$ is 1-equivalent to $(x_i)_{i<k+l}$.
(b) $\|w_i^{(0)}\|_{\mathbb{G},H} \leqslant \varepsilon$ for every $i < l$.

Let $\vec{w}_0 = (w_i^{(0)})_{i<l}$. We find $q_1 \in \mathcal{E}_{q_0,\vec{v},\vec{w}_0,\vec{x},\varepsilon/2} \cap \mathbb{G}$. Now let $(w_i^{(1)})_{i<l}$ in X_{q_1} be such that

(a′) $(v_0, \dots, v_{k-1}, w_0^{(1)}, \dots, w_{l-1}^{(1)})$ is 1-equivalent to $(\bar{x}_i)_{i<k+l}$.
(b′) $\|w_i^{(1)}\|_{H_{\mathbb{G}}} \leqslant \varepsilon/2$ for every $i < l$.
(c′) $\|w_i^{(0)} - w_i^{(1)}\|_{q_1} \leqslant \max\{\|w_i^{(0)}\|_{q_0,H}, \varepsilon/2\} \leqslant \varepsilon$ for every $i < l$.

Following this way, we can find $(q_j)_{j\in\mathbb{N}}$ in $\mathbb{G}$ and $\mathbb{Q}$-sequences $(w_i^{(j)})_{i<l}$ in X_{q_j} for each $j \in \mathbb{N}$ such that

(c_∞) $(v_0, \dots, v_{k-1}, w_0^{(j)}, \dots, w_{l-1}^{(j)})$ is 1-equivalent to $(\bar{x}_i)_{i<k+l}$ for every $j \in \mathbb{N}$.
(d_∞) $\|w_i^{(j)}\|_{\mathbb{G},H} \leqslant \varepsilon/2^j$ for every $i < l$ and every $j \in \mathbb{N}$.
(e_∞) $\|w_i^{(j)} - w_i^{(j+1)}\|_{\mathbb{G}} \leqslant \varepsilon/2^j$.

Because of the condition (e_∞) it follows that $(w_i^{(j)})_{j\in\mathbb{N}}$ is a Cauchy sequence for every $i < l$. Let $\bar{w}_i = \lim_{j\to\infty} w_i^{(j)}$ for every $i < l$. Let us check that $(\bar{w}_i)_{i<l}$ has the desired properties. From condition (d_∞) one has that $\bar{w}_i \in N_{\mathbb{G}}$ for every $i < l$, and from condition (c_∞) one has that $(v_0, \dots, v_{k-1}, \bar{w}_0, \dots, \bar{w}_{l-1})$ is 1-equivalent to $(\bar{x}_i)_{i<k+l}$.

(II): We start by proving that $\pi_{\mathbb{G}}: X_{\mathbb{G}}^{(\alpha)} \to X_{\mathbb{G},H}^{(\alpha)}$ is onto. Given $\varepsilon, \delta > 0$ $p \in \mathbb{P}_\alpha$ and $x, y \in X_p$ we define $\mathcal{D}_{p,x,y,\varepsilon}$ as the set of conditions $q \in \mathbb{P}_\alpha$ which are either incompatible with p or $q \leqslant p$ and such that there is some $z \in X_p$ such that

$$\|z-y\|_{q,H} \leqslant \varepsilon \quad \text{and} \quad \|z-x\|_q \leqslant \max\bigl\{\varepsilon, \|x-y\|_{q,H}\bigr\}.$$

Claim 3.18.2. *$\mathcal{D}_{p,x,y,\varepsilon}$ is dense.*

We check now that $\pi_{\mathbb{G}}$ is onto: Let $y \in X_{\mathbb{G},H}$. Find a sequence $(y_n)_{n\in\mathbb{N}}$ in $c_{00}(\omega_1, \mathbb{Q})$ such that $\|y_n - y\|_{\mathbb{G},H} \mapsto_n 0$ and such that $\|y_m - y_n\|_{\mathbb{G},H} \leqslant 1/2^{m+1}$ for all $m \leqslant n$. Now let $p_0 \in \mathbb{G}$ be such that $y_0 \in X_{p_0}$. Now use the claim to find $p_1 \in \mathbb{G} \cap D_{p_0,y_0,y_0,1/2}$, and $x_0 \in X_{p_1}$ such that $\|x_0 - y_0\|_{p_1} \leqslant 1/2$ (the second condition $\|x_0 - y_0\|_{p_1,H} \leqslant 1/2$ is redundant). Now let $q_1 \leqslant p_1$ in $\mathbb{G}$ be such that $y_1 \in X_{q_1}$. Now find $p_2 \in \mathcal{D}_{q_1,y_1,x_0,1/2^2}$ and $x_1 \in X_{p_2}$ such that $\|x_1 - y_1\|_{p_2,H} \leqslant 1/2^2$ and

$$\|x_1 - x_0\|_{p_2} \leqslant \max\left\{\frac{1}{4}, \|y_1 - x_0\|_{p_2,H}\right\} \leqslant \max\left\{\frac{1}{4}, \|y_1 - y_0\|_{p_2,H} + \|y_0 - x_0\|_{p_2,H}\right\} \leqslant 1.$$

Let now $q_2 \leqslant p_2$ be with $q_2 \in \mathbb{G}$ and such that $y_2 \in X_{q_2}$. Find $p_3 \in \mathcal{D}_{q_2,y_2,y_1,1/2^3}$, and $x_2 \in X_{p_3}$ such that $\|x_2 - y_2\|_{p_2,H} \leqslant 1/2^3$ and

$$\begin{aligned}\|x_2 - x_1\|_{p_3} &\leqslant \max\left\{\frac{1}{8}, \|y_2 - x_1\|_{p_3,H}\right\} \leqslant \max\left\{\frac{1}{8}, \|y_2 - y_1\|_{p_3,H} + \|y_1 - x_1\|_{p_3,H}\right\} \\ &= \max\left\{\frac{1}{8}, \|y_2 - y_1\|_{p_3,H} + \|y_1 - x_1\|_{p_2,H}\right\} \leqslant \frac{1}{2}.\end{aligned}$$

Proceeding in this way, we can find a sequence $(x_n)_n$ such that for every $n \in \mathbb{N}$ one has that $\|x_n - x_{n+1}\|_{\mathbb{G}} \leqslant 1/2^n$ and $\|x_n - y_n\|_{\mathbb{G},H} \leqslant 1/2^{n+1}$. Let then $x \in X_{\mathbb{G}}$ be the limit of the sequence $(x_n)_n$. Then

$$\pi_{\mathbb{G}}(x) = \lim_{n\to\infty} \pi_{\mathbb{G}}(x_n) = \lim_{n\to\infty} \pi_{\mathbb{G}}(x_n) + \lim_{n\to\infty} \|y_n - x_n\|_{\mathbb{G},H} = \lim_{n\to\infty} y_n = y.$$

Proof of Claim 3.18.2. Fix a condition q with $q \leqslant p$. Let $n \in \mathbb{N}$ be such that $1/n \leqslant \varepsilon$. Let now I be an interval of ordinals of cardinality n such that $D_q < I < \alpha$. Define $r = (D_q \cup I, F_r, A_q, H_q)$ where

(a) $F_r = \{\pm(h_\gamma^{(q)} + h_\gamma^{(q)}(y)\sum_{\gamma\in I} u_\gamma)\colon \gamma \in D_q\} \cup \{f + f(x)\sum_{\gamma\in I} u_\gamma\colon f \in F_q \setminus \pm H_q\} \cup \{\pm u_\gamma\colon \gamma \in I\}$.
(b) $h_\gamma^{(r)} = h_\gamma^{(q)} + h_\gamma^{(q)}(y)\sum_{\gamma\in I} u_\gamma$ for $\gamma \in D_q$ and $h_\gamma^{(r)} = u_\gamma$ for $\gamma \in I$.

It is clear that r is a basic condition such that $r \leqslant p$. The (EP) guarantees that $r \in \mathbb{P}$. Let $z := (1/n)\sum_{\gamma\in I} u_\gamma$. It is easy to check that z witnesses that $r \in \mathcal{D}_{p,x,y,\varepsilon}$. $\square$

Next, we prove that $\widetilde{\pi}_{\mathbb{G}}: X_{\mathbb{G}}^{(\alpha)}/N_{\mathbb{G}} \to X_{\mathbb{G},H}^{(\alpha)}$ is an isometry. Note that for a given $x \in X_{\mathbb{G}}^{(\alpha)}$ we have that $\|x + N_{\mathbb{G}}\| = d(x, N_{\mathbb{G}}) \geqslant \|x\|_{\mathbb{G},H}$. Our intention is to prove that, given $x \in X_{\mathbb{G}}$, there is some $y \in N_{\mathbb{G}}$ such that $\|x - y\|_{\mathbb{G}} = \|x\|_{\mathbb{G},H}$.

Let $0 < \delta \leqslant \varepsilon$, $p \in \mathbb{P}_\alpha$, and $x, y \in X_p$ such that $\|x - y\|_p \leqslant \|x\|_{p,H}$ and $\|y\|_{p,H} \leqslant \varepsilon$. We define two sets: Let $\mathcal{D}_{p,x,\varepsilon}$ be the set of all $q \perp p$ or such that $q \leqslant p$ and there is $z \in X_q$ such that $\|z\|_{q,H} \leqslant \varepsilon$ and $\|z - x\|_q \leqslant \|x\|_{q,H}$, and let $\mathcal{D}_{p,x,y,\varepsilon,\delta}$ be the set of all conditions q either $q \perp p$ or $q \leqslant p$ and there is $z \in X_p$ such that $\|z - x\|_q \leqslant \|z\|_{q,H}$, $\|z - y\|_q \leqslant \varepsilon$ and $\|z\|_{q,H} \leqslant \delta$. Then both sets are dense: For the first, fix a condition $q \leqslant p$. Let n be such that $1/n < \varepsilon$, and let I be an interval of ordinals of cardinality n with $D_q < I < \alpha$. Define $r = (D_q \cup I, F_r, A_q, H_r)$ where

(a) $F_r := \{g_f := f + f(x)\sum_{\gamma\in I} u_\gamma\colon f \in F_q \setminus \pm H_q\} \cup \pm H_q \cup \{\pm u_\gamma\colon \gamma \in I\}$.
(b) $h_\gamma^{(r)} = h_\gamma^{(q)}$ for $\gamma \in D_q$ and $h_\gamma^{(r)} = u_\gamma$ for $\gamma \in I$.

Then r is a basic condition with $r \leqslant q$ and $r \in \mathbb{P}$, because of (EP) of $\mathbb{P}$. It is easy to see that $z = (1/n)\sum_{\gamma\in I} u_\gamma$ witnesses that $r \in \mathcal{D}_{p,x,\varepsilon}$.

Now we concentrate to prove that $D_{p,x,y,\varepsilon,\delta}$ is dense: Fix a condition $q \leqslant p$. Let n be such that $1/n < \min\{\delta, \|x\|_{p,H}\}$, and let I be an interval of ordinals of cardinality n with $D_q < I$. Similarly as above, define $r = (D_q \cup I, F_r, A_q, H_r)$ where

(a) $F_r := \{g_f := f + f(y)\sum_{\gamma\in I} u_\gamma\colon f \in F_q \setminus \pm H_q\} \cup \pm H_q \cup \{\pm u_\gamma\colon \gamma \in I\}$.
(b) $h_\gamma^{(r)} = h_\gamma^{(q)}$ for $\gamma \in D_q$ and $h_\gamma^{(r)} = u_\gamma$ for $\gamma \in I$.

Then r is a basic condition with $r \leqslant q$ and $r \in \mathbb{P}$, because of (EP) of $\mathbb{P}$. Let us check that $r \in \mathcal{D}_{p,x,y,\varepsilon,\delta}$. Define $z = (1/n)\sum_{\gamma\in I} u_\gamma$. Then it is clear that $\|z\|_{r,H} = 1/n < \delta$ and $\|z - y\|_{r,H} \leqslant \max\{\max_{\gamma\in D_p} |h_\gamma^{(q)}(y)|, 1/n\} = \varepsilon$. Now, if $\gamma \in D_q$, then $|h_\gamma^{(r)}(z - x)| = |h_\gamma^{(q)}(x)| \leqslant \|x\|_{r,H}$, while if $\gamma \in I$, then $|h_\gamma^{(r)}(z - x)| = 1/n \leqslant \|x\|_{r,H}$, by the choice of n. Finally, for $f \in F_q \setminus \pm H_p$,

$$\big|g_f(z - x)\big| = \big|f(y) - f(x)\big| \leqslant \|y - x\|_q = \|y - x\|_p \leqslant \|x\|_{p,H}. \tag{24}$$

Resuming, $\|z - x\|_{r,H} \leqslant \|x\|_{p,H} = \|x\|_{r,H}$, as desired.

Now fix $x \in c_{00}(\alpha)$. Let $p_0 \in \mathbb{G}_\alpha$ with $x \in D_{p_0}$, and let $p_1 \leqslant p_0$ with $p_1 \in \mathcal{D}_{p_0,x,1/2} \cap \mathbb{G}_\alpha$. Fix $z_0 \in X_{p_1}$ such that $\|x - z_0\|_{p_1} \leqslant \|x\|_{p_1,H}$ and such that $\|z_0\| \leqslant 1/2$. Now find $p_2 \leqslant p_1$ with $p_2 \in \mathcal{D}_{p_1,x,z_0,1/2,1/4} \cap \mathbb{G}$ and fix $z_1 \in X_{p_2}$ with $\|x - z_1\|_{p_2} \leqslant \|x\|_{p_2,H}$, $\|z_1 - z_0\|_{p_2} \leqslant 1/2$ and $\|z_1\|_{p_2,H} \leqslant 1/4$, and so on. In this way, we produce a sequence $(z_n)_n$ such that for every n one has that $\|x - z_n\|_{\mathbb{G}} \leqslant \|x\|_{\mathbb{G},H}$, $\|z_n - z_{n+1}\| \leqslant 1/2^n$ and $\|z_n\|_{\mathbb{G},H} \leqslant 1/2^{n+1}$. Let $z \in X_{\mathbb{G}}$ be the limit point of $(z_n)_n$. Then $z \in N_{\mathbb{G}}$ and $\|x - z\|_{\mathbb{G}} = \lim_{n\to\infty} \|x - z_n\|_{\mathbb{G}} \leqslant \|x\|_{\mathbb{G},H}$, as desired. This finishes the proof of (I). $\square$

Given a partial ordering $\mathbb{P}$, a generic filter for it $\mathbb{G}$ and $\delta \in \Delta_{\mathbb{P}}$, it is not always the case that $A_{\mathbb{G}}^{(\delta)} = \omega_1$, even that $\langle u_\gamma;\ \gamma \in A_{\mathbb{G}}^{(\delta)}\rangle$ is dense in $X_{\mathbb{G},H}$. The next property guarantees this last fact (see the next Proposition 3.20).

Definition 3.19. We say that $\mathbb{P}$ has the *extra extension property* (EEP) if

(1) $\mathbb{P}$ has the (EP).
(2) Let $\alpha < \omega_1$ be a limit ordinal, $p \in \mathbb{P}_\alpha$, and let $\delta \in \Delta_{\mathbb{P}}$ be in the domain of $\mathbb{P}$. Then there are arbitrary large intervals $D_p < I < \alpha$ such that if q is a basic condition with

(2.1) $q \leqslant p$, $D_q = D_p \cup I$, $A_q = A_p \cup I$, and
(2.2) $h_\gamma^{(q)} = \delta u_\gamma$ for every $\gamma \in I$ and $h_\gamma^{(q)} = h_\gamma^{(p)}$ for $\gamma \in A_p^{(\delta)}$,
then q belongs to $\mathbb{P}$.

The introduction of the previous notion is justified by the following result.

Proposition 3.20. *Suppose that $\mathbb{P}$ is a forcing notion with the* (EEP). *Let $\mathbb{G}$ be a generic filter of $\mathbb{P}$. Given $\delta \in \Delta_{\mathbb{P}}$, define*

$$A_{\mathbb{G}}^{(\delta)} := \bigcup_{p\in\mathbb{G}} A_p^{(\delta)}.$$

Then for every limit $\alpha < \omega_1$ and every $\delta \in \Delta_{\mathbb{P}}$ we have that $\langle u_\gamma\rangle_{\gamma\in A_{\mathbb{G}}^{(\delta)}\cap\alpha}$ is dense in $X_{\mathbb{G}}^{(\alpha)}$ and in $X_{\mathbb{G},H}^{(\alpha)}$.

Proof. First of all, note that since $\pi_{\mathbb{G}} : X_{\mathbb{G}}^{(\alpha)} \to X_{\mathbb{G},H}^{(\alpha)}$ is onto (Theorem 3.18) the result for $X_{\mathbb{G},H}^{(\alpha)}$ follows from the result for $X_{\mathbb{G}}^{(\alpha)}$.

Now, let $p \in \mathbb{P}_\alpha$, $v \in c_{00}(D_p, \mathbb{Q})$, $\delta \in \Delta_{\mathbb{P}}$ and $\varepsilon > 0$. We define

$$\mathcal{D}_{p,v,\varepsilon,\delta} := \bigl\{q \in (\mathbb{P})_\alpha\colon \text{ either } q\perp p \text{ or } q \leqslant p \text{ and } d\bigl(v, \langle u_\gamma\rangle_{\gamma\in A_q^{(\delta)}}\bigr) \leqslant \varepsilon\|v\|_q\bigr\}.$$

Claim 3.20.1. *The set $\mathcal{D}_{p,v,\varepsilon,\delta}$ is dense.*

Let us see how to use this claim to prove the statement in the proposition. Since $\mathbb{P}$ has the (EEP), it follows that $\mathbb{G}_\alpha = \mathbb{G} \cap (\mathbb{P})_\alpha$ is a generic filter of $(\mathbb{P})_\alpha$, and $X_{\mathbb{G}}^{(\alpha)} = X_{\mathbb{G}_\alpha}$. Fix now $v \in c_{00}(\alpha, \mathbb{Q})$, and let $p \in \mathbb{G}_\alpha$ be such that $x \in X_p$. Let also $q \in \mathcal{D}_{p,v,\varepsilon,\delta} \cap \mathbb{G}_\alpha$. Since p and q are in $\mathbb{G}$ they are compatible, so it follows that $q \leqslant p$ and $d(v, \langle\sigma_q^{(n)}\rangle) \leqslant \|v\|_p$. Finally, using that $\sigma_q^{(n)} \subseteq \sigma_{\mathbb{G}}^{(n)}$, one has that $d(v, \langle\sigma_{\mathbb{G}}^{(n)}\rangle) \leqslant \varepsilon\|v\|_{\mathbb{G}}$, and we are done.

Proof of Claim 3.20.1. Fix $p_0 \in \mathbb{P}_\alpha$. If $p_0\perp p$ then $p_0 \in \mathcal{D}_{p,v,\varepsilon,\delta}$ and we are done. So, we suppose otherwise that p_0 and p are compatible in $\mathbb{P}_\alpha$, and we fix $p_1 \in \mathbb{P}_\alpha$ such that $p_1 \leqslant p_0, p$. Without loss of generality we assume, after corresponding normalization, that $\|v\|_p = 1$. Let $d = \#(A_{p_1}^{(\delta)})$, and $X := \langle u_\gamma\rangle_{\gamma\in A_{p_1}^{(\delta)}}$. Since the set $H_{p_1}^{(\delta)} := \{h_\gamma^{(p_1)}\colon \gamma \in A_{p_1}^{(\delta)}\}$ has cardinality d and it separates points of X it follows that there is some $\bar{v} \in X$ such that

$$h_\gamma^{(p_1)}(\bar{v}) = h_\gamma^{(p_1)}(v) \quad \text{for all } \gamma \in A_{p_1}^{(\delta)}. \tag{25}$$

Now if $v = \bar{v} \in \langle u_\gamma\rangle_{\gamma\in A_{p_1}^{(\delta)}}$, then $p_1 \in \mathcal{D}_{p,v,\varepsilon,\delta}$, $p_1 \leqslant p_0$ and we are done. Now suppose that $v \neq \bar{v}$. Let $w = v - \bar{v}$, and let $k \in \mathbb{N}$ be such that $k \cdot \varepsilon \geqslant \|w\|_{p_1}$. We use property (2.2) of $\mathbb{P}$ to fix an interval $D_{p_1} < I < \alpha$ with $\#(I) \geqslant k$. Now for each $f \in F_{p_1}$, define

$$g_f := f + \frac{f(w)}{\|w\|_{p_1}} \sum_{\gamma\in I} u_\gamma.$$

Let $q = (D_q, F_q, A_q, H_q)$ be defined as follows:

$$F_q := \{g_f\colon\ f \in F_{p_1}\} \cup \{\pm\delta u_\gamma\colon\ \gamma \in I\},$$

$A_q = A_{p_1} \cup I$, and for $\gamma \in \sigma_q$ we have

$$h_\gamma^{(q)} := \begin{cases} g_{h_\gamma^{(p)}} & \text{if } \gamma \in A_{p_1}, \\ \delta u_\gamma & \text{if } \gamma \in I. \end{cases}$$

Note that $h_\gamma^{(q)} \restriction I = 0$ for every $\gamma \in A_{p_1}^{(\delta)}$. Then q is in $\mathbb{P}$ because $\mathbb{P}$ has the (EEP). Let us now prove that $q \in \mathcal{D}_{p,v,\varepsilon,\delta}$. First of all, we see that

$$\left\| w - \frac{\|w\|_{p_1}}{k \cdot \delta} \sum_{i<k} x_i \right\|_q \leqslant \varepsilon: \tag{26}$$

For $f \in F_{p_1}$ we have

$$g_f\left(w - \frac{\|w\|_{p_1}}{k \cdot \delta} \sum_{\gamma \in I} u_\gamma\right) = f(w) - \frac{\|w\|_{p_1}}{k \cdot \delta} \frac{f(w)}{\|w\|_{p_1}} \sum_{\gamma \in I} \delta = f(w) - \|w\|_{p_1} \frac{f(w)}{\|w\|_{p_1}} = 0.$$

For $\gamma \in I$ we have

$$\left| \delta u_\gamma \left(w - \frac{\|w\|_{p_1}}{k \cdot \delta} \sum_{i<k} x_i \right) \right| \leqslant \frac{\|w\|_{p_1} \cdot \delta}{k \cdot \delta} \leqslant \varepsilon.$$

It follows from (26) that

$$d\left(v, \left\langle \sigma_q^{(n)} \right\rangle\right) \leqslant \left\| v - \bar{v} - \frac{\|w\|_{p_1}}{k \cdot \delta} \sum_{\gamma \in I} u_\gamma \right\|_q \leqslant \varepsilon,$$

so $q \in \mathcal{D}_{p,v,\varepsilon,\delta}$ we are done. $\square$

3.3. Amalgamation and types

So far, we did not discuss the density of the generic spaces associated to the basic forcing notion. We address this question in this section, where the notion of *amalgamation* of conditions is presented. In retrospective, we mention that while arguments involving dense subsets of $\mathbb{P}$ seen above will give us local or separable properties of the corresponding generic spaces, arguments involving amalgamations of conditions are the keys to understand the global properties of non-separable subspaces of generic spaces.

Definition 3.21. Recall that $p, q \in \mathbb{P}$ are *compatible* if there is $r \in \mathbb{P}$ of p and q, such that $r \leqslant p$ and $r \leqslant q$. Otherwise, we say that p and q are *incompatible*. In general we say that a finite set of conditions $N \subseteq \mathbb{P}$ is compatible if there is some $p \in \mathbb{P}$ such that $p \leqslant q$ for every $q \in N$.

Note that a generic filter $\mathbb{G}$ of a given forcing notion $\mathbb{P}$ is in particular a set of pairwise compatible conditions of $\mathbb{P}$, so it is not surprising that the notion of compatibility plays a fundamental role in the understanding of the generic space $X_{\mathbb{G}}$. But not only for $X_{\mathbb{G}}$. A priori, the generic filter $\mathbb{G}$ can construct a map from ω onto ω_1, so the generic space $X_{\mathbb{G}}$ will be automatically a *separable* Banach space. This is one thing that we want to prevent. A large variety of conditions on $\mathbb{P}$ could be used to show that this does not happen. For the sake of completeness, we recall two of them.

Definition 3.22. A partial ordering $\mathbb{P}$ has the *countable chain condition* (*ccc* in short) if every antichain (i.e. a subset of $\mathbb{P}$ consisting on pairwise incompatible elements) is countable. The partial ordering $\mathbb{P}$ has the *Shanin property* if every uncountable subset $\mathcal{A} \subseteq \mathbb{P}$ has an uncountable subset $\mathcal{A}_0 \subseteq \mathcal{A}$ with the property that for every finite subset $\mathcal{F} \subseteq \mathcal{A}_0$ there is $p \in \mathbb{P}$ such that $p \leqslant q$ for every $q \in \mathcal{F}$.

It is clear that the Shanin property is stronger than the countable chain condition. We recall also the well-known fact that if $\mathbb{P}$ satisfies the countable chain condition then its forcing extension preserves all cardinals (see for example Kunen [18]). The Shanin property is one of the strengthenings of the countable chain condition that is productive in the sense that if $\prod_i P_i$ is the finitely-supported product of a family $\{P_i\}_i$ of partial orderings having the Shanin property, then so does the product. Since all the partial orderings we are going to present have the Shanin property, the previous fact says that we can have all our examples simultaneously.

Let us now analyze uncountable sequences $(p_\alpha)_{\alpha<\omega_1}$ of basic conditions $p_\alpha = (D_\alpha, F_\alpha, A_\alpha, H_\alpha)$. Since each D_α is finite, the Δ-System Lemma given an uncountable subset $\Gamma \subseteq \omega_1$ such that $(D_\alpha)_{\alpha\in\Gamma}$ forms a Δ-*system*, i.e., there is some finite set R such that for every $\alpha < \beta$ one has that

$$N := \#(D_\alpha) = \#(D_\beta), \qquad D_\alpha \cap D_\beta = R \quad \text{and} \quad R < D_\alpha \setminus R < D_\beta \setminus R. \tag{27}$$

In addition, and because of practical purposes, we also require that:

$$\text{For every } \alpha < \omega_1 \text{ one has that } \bigl[\max R, \min(D_\alpha \setminus R)\bigl[\text{ is infinite.} \tag{28}$$

Now for each $\alpha \in \Gamma$, let $\theta_\alpha : D_\alpha \to |\#(D_\alpha)| = N$ be the unique order-preserving bijection. This extends naturally to an isometry onto $\theta_\alpha : c_{00}(D_p) \to c_{00}(N)$. Using this isometry, we can naturally define the type t_α of p_α as the basic condition $(N, \theta_\alpha(F_\alpha), \theta_\alpha(A_\alpha)), \theta_\alpha(H_\alpha))$ (see Definition 3.23 below for explicit details). Since each $F_\alpha \subseteq c_{00}(D_\alpha, \mathbb{Q})$ and is finite, a simple counting argument gives that there is a further uncountable subset $\Gamma_0 \subseteq \Gamma$ such that $t := t_\alpha = t_\beta$ for every $\alpha, \beta \in \Gamma_0$. So, in order to show that $\mathbb{P}_b$ has the Shanin property we have to analyze Δ-systems $(p_\alpha)_{\alpha<\omega_1}$ of type t, and see whether finitely many conditions $p_{\alpha_0}, \dots, p_{\alpha_n}$ are compatible. We will see later that this is the case. Now we define properly the concepts we have just introduced.

Definition 3.23. Given a basic condition $p = (D_p, F_p, A_p, H_p)$, and points $x_0, \dots, x_{n-1} \in c_{00}(D_p, \mathbb{Q})$ the *type of* $(p, (x_i)_{i<n})$, is the pair

$$\bigl(\theta_p(p), \bigl(\theta_p(x_i)\bigr)_{i<n}\bigr)$$

where $\theta_p : D_p \to |D_p|$ is the unique order-preserving bijection $\theta_p : D_p \to |D_p|$, and $\theta_p(p)$ is the θ_p-spread of p (see Definition 3.8).

Definition 3.24. A sequence $\vec{p} = (p_\alpha, (v_\alpha^{(i)})_{i<n})_{\alpha<\kappa}$ of elements of $(\mathbb{P} \times c_{00}(\omega_1, \mathbb{Q})^n)^\kappa$ is called a Δ-*system* if:

(1) $v_\alpha^{(i)} \in c_{00}(D_{p_\alpha}, \mathbb{Q})$ for every $\alpha < \kappa$, and every $i < n$,
(2) $(D_{p_\alpha})_{\alpha<\kappa}$ is a Δ-system of sets (in particular we assume that $[\max R, \min D_{p_\alpha} \setminus R]$ is infinite for every $\alpha < \kappa$),
(3) for every $\alpha < \beta < \kappa$ one has that $(\theta_{p_\alpha}(p_\alpha), (\theta_{p_\alpha}(v_\alpha^{(i)}))_{i<n}) = (\theta_{p_\beta}(p_\beta), (\theta_{p_\beta}(v_\beta^{(i)}))_{i<n})$.

Given such sequence $\vec{p}$, we say that

$$\mathrm{tp}(\vec{p}) = \bigl(\theta_0(p_0), \bigl(\theta_{p_0}\bigl(v_0^{(i)}\bigr)\bigr)_{i<n}\bigr)$$

is the *type* $\mathrm{tp}(\vec{p})$ of $\vec{p}$. The *root* $\mathrm{R}(\vec{p})$ of $\vec{p}$ is the root of $(D_{p_\alpha})_{\alpha<\omega_1}$, i.e.,

$$\mathrm{R}(\vec{p}) = D_{p_0} \cap D_{p_1}.$$

We say that $x \in X_{p_\alpha}$ is the *twin* of $y \in X_{p_\beta}$ if $\theta_{p_\alpha}(x) = \theta_{p_\beta}(y)$.

We have proved in the introduction of this subsection the following.

Proposition 3.25. *For every $(p_\alpha, (v_\alpha^{(i)})_{i<n})_{\alpha<\omega_1} \in (\mathbb{P} \times c_{00}(\omega_1, \mathbb{Q})^n)^\kappa$ there is an uncountable $\Gamma \subseteq \omega_1$ such that $(p_\alpha, (v_\alpha^{(i)})_{i<n})_{\alpha\in\Gamma}$ is a Δ-system.* $\square$

Suppose that p and q are two basic conditions which are compatible by r, i.e. $r \leqslant p, q$. Then there are metric constraints:

(a) The norm $\|\cdot\|_r$ extends both $\|\cdot\|_p$ and $\|\cdot\|_q$ and the same for $\|\cdot\|_{r,H}$. In particular,
  (a.1) the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ coincide in the intersection space $c_{00}(D_p \cap D_q)$ of X_p and X_q, and the same for $\|\cdot\|_{p,H}$ and $\|\cdot\|_{q,H}$.

There are also "structural constrains":

(b) $A_p \cap D_q = A_q \cap D_p = A_p \cap A_q$ and for every $\gamma \in D_p \cap D_q$ one has that

$$h_\gamma^{(p)} \restriction D_q = h_\gamma^{(q)} \restriction D_p. \tag{29}$$

(c) For every $\gamma \in D_p \cap D_q$ one has that

$$h_\gamma^{(r)} \restriction (D_p \cup D_q) = h_\gamma^{(p)} \vee h_\gamma^{(q)}. \tag{30}$$

Definition 3.26. We say that $p_0, \dots, p_n \in \mathbb{P}$ are *pre-compatible* if there is some R such that $D_{p_i} \cap D_{p_j} = R$ for every $i < j \leqslant n$ and (a.1) and (b) above hold for p_i, p_j, for every $i, j \leqslant n$. A condition r is a *pre-amalgamation* of pre-compatible $p_0, \dots, p_n$ if

(1) $D_r = \bigcup_{i \leqslant n} D_{p_i}$ and $A_r = \bigcup_{i \leqslant n} A_{p_i}$.
(2) For every $i, j \leqslant n$, (a), (b) and (c) above hold for p_i, p_j and r.

It is clear that finite Δ-systems are always pre-compatibles. We will see soon that indeed they are indeed compatible in the basic forcing $\mathbb{P}_{\mathrm{basic}}$. In general, being pre-compatible does not suffice to be compatible, even for the basic forcing notion $\mathbb{P}_{\mathrm{basic}}$. This is a consequence of the fact that in general the relative position of the supports plays an important role in the compatibility, as one can see in the properties of $h_\gamma^{(p)}$: Suppose that p and q are pre-compatible, and let $R = D_p \cap D_q$. Suppose that there are $\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3$ with $\gamma_1, \gamma_3 \in R$, $\gamma_0 \in D_p \setminus R$ and $\gamma_1 \in D_q \setminus R$. If p and q were compatible, and $r \leqslant p, q$, there would be some $h \in \mathrm{conv}_{\mathbb{Q}}(\pm H_p)$ such that $h_{\gamma_1}^{(r)} \upharpoonright (D_p \cup D_q) = h_\gamma^{(q)} \vee h$. In particular, $(h)_{\gamma_0} = (h_{\gamma_1}^{(r)})_{\gamma_0} = 0$, and the existence of h is not guaranteed by the fact that p and q are simply pre-compatible, unless, for example R is initial part of both D_p and D_q, and then the previous situation does not occur.

Proposition 3.27. *Every finite Δ-system $(p_i)_{i<k}$ of basic conditions is compatible in $\mathbb{P}_{\mathrm{basic}}$, i.e. there is some basic condition $p \leqslant p_i$ for every $i < k$. Hence, the basic forcing notion $\mathbb{P}_{\mathrm{basic}}$ has the Shanin property.*

Proof. Fix a Δ-system $(p_i)_{i<k}$ of type $t = (N, F, A, H)$ and root R, and let $\theta_i : N \to D_{p_i}$ be the corresponding order-preserving bijections. It should be clear that $(p_i)_{i<k}$ are pre-compatible. We define now a pre-amalgamation $p = (D_p, F_p, A_p, H_p)$ as follows:

(a) F_p is the minimal symmetric subset of $c_{00}(D_p)$ containing:
  (a.1) $\bigvee_{i<k} \theta_i(g)$ for every type $g \in F \setminus \pm H$ or for every type g of some $h_\gamma^{(p_0)}$ with $\gamma \in R$.
  (a.2) $H_\gamma^{(p_i)}$ for every $i < k$ and every $\gamma \in D_i \setminus R$.
(b) For every $i < k$ and $\gamma \in D_{p_i}$, we define:

$$h_\gamma^{(p)} := \begin{cases} \bigvee_{j<k} \theta_j(g) & \text{if } \gamma \in R, \\ h_\gamma^{(p_i)} & \text{if } R < \gamma, \end{cases}$$

where g is the type of $h_\gamma^{(p_i)}$.

The condition p above is a sort of minimal amalgamation of $(p_i)_{i<k}$. We call it the *basic amalgamation* of $(p_i)_{i<k}$. It is easy to see that the result p of the amalgamation is a basic condition and that $p \leqslant p_i$ for every $i < k$. $\square$

Definition 3.28. We say that a forcing notion $\mathbb{P}$ has the *amalgamation property* (AMP) if whenever $(p_i)_{i<n}$ is a Δ-system of conditions of $\mathbb{P}$ then its basic amalgamation p belongs to $\mathbb{P}$.

Proposition 3.29. *Suppose that $\mathbb{P}$ has* (AMP). *Then,*

(1) $\mathbb{P}$ *has the Shanin property.*
(2) $\mathbb{P}_H$ *has* (AMP). $\square$

Corollary 3.30. *If $\mathbb{P}$ has* (AMP) *and* (EP)*, then every generic space is non-separable* $\square$

The following strengths the (AMP) and it will be used, among some other things, to control operators on the generic spaces. It is somehow a mix between (AMP) and (EP).

Definition 3.31. We say that a forcing notion $\mathbb{P}$ has the *extended amalgamation property* (EAMP) when $\mathbb{P}_H \subseteq \mathbb{P}$ and the following happens: Let $(p_i)_{i<k}$ be a Δ-system of conditions of $\mathbb{P}$ with root R and type (N, F, A, H), $\theta_i : N \to D_i$ be the corresponding order-preserving mapping for every $i < k$. Suppose that $p = (D_p, F_p, A_p, H_p)$ is a basic condition such that:

(a) $p \leqslant p_i$ for every $i < k$ and $A_p = \bigcup_{i<k} A_i$.
(b) For every $\gamma \in R$ one has that $h_\gamma^{(p)} = \bigvee_{i<k} \theta_i(g)$, where g is the type of $h_\gamma^{(p_0)}$.
(c) For every $\gamma \in D_i \setminus R$ one has that $h_\gamma^{(p)} = h_\gamma^{(p_i)}$.

Then $p \in \mathbb{P}$.

It is clear that the basic amalgamation is a particular case of p defined above.

3.4. Unavoidable configurations. Types

The following notion of configuration is the key tool for our understanding the non-separable structure of the generic spaces. It is a weakening of the notion of first order formula from mathematical logic adapted to the context of our interest here. Configurations will be used to reformulate the important Forcing Theorem to our context here (Theorem 3.36).

Definition 3.32. Let $\mathcal{V}$ be a $\mathbb{Q}$-f.d. space with two $\mathbb{Q}$-norms on it, $\|\cdot\|$ and $\|\cdot\|_H$, and let $v_0, \dots, v_{n-1}$ be a sequence of $\mathbb{Q}$-points of V. We let a *configuration* $\mathfrak{P}(v_0, \dots, v_{n-1})$ be a property describing an action of some rational combinations of the vectors $v_0, \dots, v_{n-1}$ on the dual balls of $(V, \|\cdot\|)$ and/or $(V, \|\cdot\|_H)$. While this can be properly defined using tools from mathematical logic, we choose to explain what we mean by this in every specific example that we give below.

Given such configuration $\mathfrak{P}(v_0, \dots, v_{n-1})$, a $\mathbb{Q}$-f.d. space $\mathcal{X} = (X, \|\cdot\|_X)$ and $\mathbb{Q}$-points $(x_i)_{i<n}$ of X we say that $\mathfrak{P}(x_0, \dots, x_{n-1})$ *holds* (in $\mathcal{X}$) if the natural *interpretation* of the configuration $\mathfrak{P}$ in $\mathcal{X}$ with respect to the points $x_0, \dots, x_{n-1}$ is true in $\mathcal{X}$.

We write a configuration $\mathfrak{P}(v_0^{(0)}, \dots, v_{n-1}^{(0)}, \dots, v_0^{(k)}, \dots, v_{n-1}^{(k)})$ by $\mathfrak{P}(\vec{v}_0, \dots, \vec{v}_k)$ because we want to distinguish the roles of the variables.

We give some examples to help us understand the previous concept.

Examples 3.33. (a) Given rational numbers $\lambda_0, \dots, \lambda_{n-1}$ and λ, the inequality

$$\left\| \sum_{i<n} a_i v_i \right\| \geqslant \lambda \tag{31}$$

is an example of a configuration that we call a *metric configuration*.

(b) Given a $\mathbb{Q}$-basis $((e_i)_{i<n}, \|\cdot\|_G)$ and a rational number $K \geqslant 1$, the standard inequalities describing that

$$(v_i)_{i<n} \text{ is } K\text{-equivalent to } \big((e_i)_{i<n}, \|\cdot\|_G\big) \tag{32}$$

is another example of a metric configuration.

(c) Given a rational $\varepsilon > 0$ and an integer m, the condition

$$\text{for every } f_0, \dots, f_m \in B_{V^*} \text{ there is } 1 \leqslant i \leqslant n-1 \text{ such that } \max_{j \leqslant m} \big| f_j(v_0) - f_j(v_i) \big| \leqslant \varepsilon$$

is a configuration.

Definition 3.34. Let $\mathbb{G}$ be a generic filter for a forcing notion $\mathbb{P}$. We say that a configuration $\mathfrak{P}(\vec{v}_0, \dots, \vec{v}_k)$, $\vec{v}_i = (v_0^{(i)}, \dots, v_{n-1}^{(i)})$ for $i \leqslant k$, is *unavoidable* for the finite sequence $(y_\alpha^{(0)})_{\alpha<\omega_1}, \dots, (y_\alpha^{(k)})_{\alpha<\omega_1}$ of ω_1-sequences of points of $c_{00}(\omega_1, \mathbb{Q})$ if for every uncountable $\Gamma \subseteq \omega_1$ there are $\xi_0 < \dots < \xi_{n-1}$ in Γ such that $\mathfrak{P}((y_{\alpha_{\xi_i}}^{(0)})_{i<n}, \dots, (y_{\alpha_{\xi_i}}^{(k)})_{i<n})$ holds in $X_{\mathbb{G}}$., i.e. it holds in $c_{00}(\bigcup_{i<n, j<k} \operatorname{supp}(y_{\alpha_{\xi_i}}^{(j)}), \|\cdot\|_{\mathbb{G}}, \|\cdot\|_{\mathbb{G},H})$.

A configuration $\mathfrak{P}(\vec{v}_0, \dots, \vec{v}_k)$ is unavoidable in $X_{\mathbb{G}}$ if it is unavoidable for any finite sequence of one-to-one ω_1-sequences of points of $X_{\mathbb{G}} \cap c_{00}(\omega_1, \mathbb{Q})$. Finally, we say that the configuration $\mathfrak{P}(\vec{v}_0, \dots, \vec{v}_k)$ is *unavoidable for* $\mathbb{P}$ if it is unavoidable in every generic space $X_{\mathbb{G}}$ of $\mathbb{P}$.

The following fact follows easily.

Proposition 3.35. *Let $X_{\mathbb{G}}$ be a generic space for a forcing notion $\mathbb{P}$, let $\mathfrak{P}(\vec{v}_0, \dots, \vec{v}_k)$ be a configuration, and let $\vec{x}_0 = (x_j^{(0)})_{j<n}, \dots, \vec{x}_k = (x_j^{(k)})_{j<n}$ be in $c_{00}(\omega_1, \mathbb{Q})$. Then the following are equivalent*:

(a) *$\mathfrak{P}(\vec{x}_0, \dots, \vec{x}_k)$ holds in $X_{\mathbb{G}}$.*
(b) *There is $p \in \mathbb{G}$ with $\vec{x}_0, \dots, \vec{x}_k$ in $c_{00}(D_p)$ such that $\mathfrak{P}(\vec{x}_0, \dots, \vec{x}_k)$ holds in X_p.*
(c) *For every $p \in \mathbb{G}$ if $\vec{x}_0, \dots, \vec{x}_k$ are in $c_{00}(D_p)$, then $\mathfrak{P}(\vec{x}_0, \dots, \vec{x}_k)$ holds in X_p.* $\square$

The following is an important result that will help us reduce the study of properties of generic spaces of some forcing notion $\mathbb{P}$ to the study of finite amalgamations that are possible in $\mathbb{P}$.

Theorem 3.36. *Let $\mathfrak{P}((v_j^{(0)})_{j<n}, \dots, (v_j^{(k)})_{j<n})$ be a configuration, and let $\mathbb{P}$ be an arbitrary forcing notion. Suppose that*:

(a) *Whenever $(p_j, (v_j^{(i)})_{i \leqslant k})_{j<n}$ is a Δ-sequence in $\mathbb{P}$ there is a condition $p \in \mathbb{P}$ such that*
  (a.1) *$p \leqslant p_j$ for every $j < n$ and*
  (a.2) *$\mathfrak{P}((v_j^{(0)})_{j<n}, \dots, (v_j^{(k)})_{j<n})$ holds in X_p.*

Then

(b) *$\mathfrak{P}((v_j^{(0)})_{j<n}, \dots, (v_j^{(k)})_{j<n})$ is unavoidable for $\mathbb{P}$.*

If in addition $\mathbb{P}$ *is hereditary, spreading and it satisfies the countable chain condition, then* (a) *and* (b) *are equivalent.*

Proof. Suppose first that (a) holds. Fix a generic filter $\mathbb{G}$ for $\mathbb{P}$. Let $\vec{x}_0, \dots, \vec{x}_k$ be an uncountable sequence of points of $c_{00}(\omega_1, \mathbb{Q})$, $\vec{x}_i = (x_\alpha^{(i)})_{\alpha<\omega_1}$, $i \leqslant k$. Let $\sigma^{(0)}, \dots, \sigma^{(k)}$ be $\mathbb{P}$-names such that for every $i \leqslant k$ one has that $(\sigma^{(i)})_{\mathbb{G}} = \vec{x}_i$. Let $p \in \mathbb{G}$ be any condition such that

$$p \Vdash \text{“}\sigma^{(i)} \text{ is a one-to-one } \omega_1\text{-sequence of points of } c_{00}(\omega_1, \mathbb{Q}) \text{ for every } i \leqslant k\text{”}. \tag{33}$$

Let $\mathcal{D}_p$ be the set of all the conditions $q \in \mathbb{P}$ such that either $q \perp p$ or $q \leqslant p$ and there are ordinals $\alpha_0 < \dots < \alpha_{n-1}$ and points $(v_j^{(i)})_{i\leqslant k, j<n}$ in $c_{00}(D_q, \mathbb{Q})$ such that

(I) q forces that the value of $\sigma^{(i)}$ in α_j is $v_j^{(i)}$ for every $i \leqslant k$ and $j < n$, and
(II) the configuration $\mathfrak{P}((x_j^{(0)})_{j<n}, \dots, (x_j^{(k)})_{j<n})$ holds in X_q.

Claim 3.36.1. *The set* $\mathcal{D}_p$ *is dense.*

Let us see how to use this claim to conclude the proof. Since $\mathcal{D}_p$ is dense and $p \in \mathbb{G}$, it follows that there is some $q \leqslant p$ such that (I) and (II) above holds. By (I), one has that $x_{\alpha_j}^{(i)} = v_j^{(i)}$ for all $i \leqslant k$ and $j < n$, and then (II) means that $\mathfrak{P}((x_{\alpha_j}^{(0)})_{j<n}, \dots, (x_{\alpha_j}^{(k)})_{j<n})$ holds, as desired. So, it rests to show the previous claim.

Proof of Claim 3.36.1. Let $\bar{p} \leqslant p$ be an arbitrary condition. Since $\bar{p}$ also forces that $\sigma^{(i)}$ is an ω_1-sequence of points of $c_{00}(\omega_1, \mathbb{Q})$ for every $i \leqslant k$, we can find $(p_\alpha, (v_\alpha^{(i)})_{i\leqslant k})_{\alpha<\omega_1}$ such that for every $\alpha < \omega_1$ one has that

(3) $p_\alpha \in \mathbb{P}$ is such that $p_\alpha \leqslant p$.
(4) $(v_\alpha^{(i)})_{i\leqslant k}$ is a sequence of points in $c_{00}(D_{p_\alpha}, \mathbb{Q})$, and
(5) p_α forces that the value of $\sigma^{(i)}$ in α is $v_\alpha^{(i)}$ for every $i \leqslant k$.

By a simple use of the Δ-System Lemma, there is an uncountable $\Gamma \subseteq \omega_1$ such that $(p_\alpha, (v_\alpha^{(i)})_{i\leqslant k})_{\alpha\in\Gamma}$ is a Δ-system of type $t = (N, F, A, H, (v^{(i)})_{i\leqslant k})$ and root R. In particular, if $\alpha_0 < \dots < \alpha_{n-1}$ are in Γ, then, by (a), there is some $q \in \mathbb{P}$ such that $q \leqslant p_{\alpha_j}$ for every $j < n$ and such that the configuration $\mathfrak{P}((v_{\alpha_0}^{(i)})_{i\leqslant n}, \dots, (v_{\alpha_{n-1}}^{(i)})_{i\leqslant n})$ holds in X_q. So $q \leqslant \bar{p}$ and $q \in \mathcal{D}_p$, as desired. $\square$

For the implication (b) $\to$ (a), we assume that $\mathbb{P}$ is hereditary, spreading and it satisfies the countable chain condition, and we suppose that (b) does not hold, and let $(p_j, (v_j^{(i)})_{i\leqslant k})_{j<n}$ be a witness of it of type $t = (N, F, A, H, (v^{(i)})_{i\leqslant k})$ and root R. Using that $\mathbb{P}$ is spreading, we can extend the finite Δ-system $(p_j, (v_j^{(i)})_{i\leqslant k})_{j<n}$ to a Δ-system $(p_\alpha, (v_\alpha^{(i)})_{i\leqslant k})_{\alpha<\omega_1}$ with the same type t and root R. Let $r = p_0 \upharpoonright R$, which is a condition in $\mathbb{P}$ because the basic forcing is hereditary. Now we use the following well-known fact true about any forcing notion satisfying the countable chain condition.

Claim 3.36.2. *For some generic filter $\mathbb{G}$ of $\mathbb{P}$, the set $\{\alpha < \omega_1\colon\ p_\alpha \in \mathbb{G}\}$ is uncountable.*

Fix such generic filter $\mathbb{G}$. Without loss of generality, after re-enumeration, we may assume that $p_\alpha \in \mathbb{G}$ for all $\alpha < \omega_1$. We claim that for no $\alpha_0 < \cdots < \alpha_{n-1} < \omega_1$ the property $\mathfrak{P}((x_{\alpha_0}^{(i)})_{i \leqslant k}, \ldots, (x_{\alpha_{n-1}}^{(i)})_{i \leqslant k})$ holds: Suppose otherwise that $\mathfrak{P}((x_{\alpha_0}^{(i)})_{i \leqslant k}, \ldots, (x_{\alpha_{n-1}}^{(i)})_{i \leqslant k})$ holds in $X_{\mathbb{G}}$ for some $\alpha_0 < \cdots < \alpha_{n-1} < \omega_1$. Using Proposition 3.35, we fix a condition $p \in \mathbb{G}$ such that for every $j < n$ the sequence $(x_{\alpha_j}^{(i)})_{i \leqslant k}$ is in $c_{00}(D_p, \mathbb{Q})$ and such that $\mathfrak{P}((x_{\alpha_0}^{(i)})_{i \leqslant k}, \ldots, (x_{\alpha_{n-1}}^{(i)})_{i \leqslant k})$ holds in X_p. Using that $\mathbb{G}$ is a filter, we find $q \in \mathbb{G}$ such that $q \leqslant p, q_{\alpha_0}, \ldots, q_{\alpha_{n-1}}$. It follows from Proposition 3.35 that $\mathfrak{P}(x_{\alpha_0}, \ldots, x_{\alpha_{n-1}})$ also holds in X_q. We now use our assumption on Δ-systems that for each $j < n$ one has that

$$\big[\max R, \min(D_{p_j} \setminus R)\big] \text{ is infinite} \tag{34}$$

to find an order-preserving bijection $\theta : D_q \to \theta(D_q)$ such that $\theta \restriction D_{p_{\alpha_j}} = \theta_j$ and such that $\theta(\gamma + 1) = \theta(\gamma) + 1$ for every $\gamma \in D_q$. Notice that $\theta(x_{\alpha_j}^{(i)}) = x_j^{(i)}$ for every $i \leqslant k$ and every $j < n$. Let $\bar{q} = \theta(q)$. Since $\mathbb{P}$ is spreading, it follows that $\bar{q} \in \mathbb{P}$, and $\bar{q} \leqslant \theta(p_{\alpha_j}) = p_j$. Finally, since θ extends to a natural isometry from X_q to $X_{\bar{q}}$ it follows that the configuration $\mathfrak{P}((\theta(x_{\alpha_0}^{(i)}))_{i \leqslant k}, \ldots, (\theta(x_{\alpha_{n-1}}^{(i)}))_{i \leqslant k})$ holds in $X_{\bar{q}}$, i.e. $\mathfrak{P}((x_j^{(i)})_{i \leqslant k}, \ldots, (x_j^{(i)})_{i \leqslant k})$ holds in $X_{\bar{q}}$. $\square$

3.5. *A first application*

In this subsection we show how easily we get spaces that are analogous to those constructed by Kunen (see [31]), Shelah [37] and Todorcevic [38] (see the discussion in the Introduction, and [5,12]). More precisely, as the first application of Theorem 3.36, we examine the existence of uncountable biorthogonal sequences in generic spaces over our basic forcing. So, we fix a generic filter $\mathbb{G}$ of the basic forcing $\mathbb{P}_{\mathrm{basic}}$.

Proposition 3.37. *The generic space $X_{\mathbb{G}}$ does not have uncountable biorthogonal sequences.*

Proof. Otherwise, there is some uncountable normalized sequence $(x_\alpha)_{\alpha < \omega_1}$ consisting of points of $c_{00}(\omega_1, \mathbb{Q})$ and there exists some integer n such that

$$\left\| x_\alpha - \frac{1}{n} \sum_{\beta \in s} x_\beta \right\|_{\mathbb{G}} \geqslant \frac{2}{n} \quad \text{for every } \alpha < s \text{ with } \#s = n. \tag{35}$$

Let $\mathfrak{P}(v_0, \ldots, v_n)$ be the metric configuration: Either there is some $i \leqslant n$ such that $\|v_i\| \neq 1$, or else

$$\left\| v_0 - \frac{1}{n} \sum_{i=1}^{n} v_i \right\| \leqslant \frac{1}{n} \max_{i \leqslant n} \|v_i\|. \tag{36}$$

Our intention is to prove that this configuration is unavoidable, so (35) is impossible. We use Theorem 3.36: Let $(p_i, v_i)_{i \leqslant n}$ be a Δ-system with root R and type $t = (N, F, \sigma, H, v)$. For each $i \leqslant n$ let $\theta_i : N \to D_i$ be the order-preserving bijection. Define the amalgamation $p = (\bigcup_{i \leqslant n} D_i, F_p, \bigcup_{i \leqslant n} A_i, H_p)$ as follows:

(a) F_p is the minimal symmetric subset of $c_{00}(D_p, \mathbb{Q})$ containing:
   (a.1) $\bigvee_{i<n} \theta_i(g)$ for every $g \in F$.
   (a.2) $h_\gamma^{(p_i)}$ for every $1 \leqslant i \leqslant n$ and every $\gamma \in D_i \setminus R$.
(b) Given $i \leqslant n$ and $\gamma \in D_i$, we define

$$h_\gamma^{(p)} := \begin{cases} \bigvee_{j<n} \theta_j(h) & \text{if } i = 0\text{, and where } h \in F \text{ is the type of } h_\gamma^{(p_0)},\\ h_\gamma^{(p_i)} & \text{otherwise.} \end{cases}$$

Then clearly p is a basic condition such that $p \leqslant p_i$ for every $i < n$. We check that the configuration $\mathfrak{P}(v_0, \dots, v_{n-1})$ holds in X_p: If $\|v\|_t \neq 1$, then clearly $\|v_0\|_p = \|v\|_t \neq 0$. Otherwise, suppose that $\|v\|_t = 1$, and we work to prove (36): Fix $f \in F_p$. Suppose that $f = \bigvee_{i \leqslant n} \theta_i(g)$ is as in (a.1). Then

$$f\left(v_0 - \frac{1}{n}\sum_{i=1}^{n} v_i\right) = f(v) - \frac{1}{n} n f(v) = 0.$$

If $f = h_\gamma^{(p_i)}$ is as in (a.2), then

$$\left|f\left(v_0 - \frac{1}{n}\sum_{i=1}^{n} v_i\right)\right| = \frac{1}{n}\left|f(v_i)\right| \leqslant \frac{1}{n}\|v_i\|$$

where $1 \leqslant i \leqslant n$ is such that $f \in F_i$. $\square$

We shall see later that this space has the stronger property that the space of continuous functions on the dual ball is hereditarily Lindelöf in all finite powers relative to its topology of pointwise convergence (see Remark 8.15).

4. Mazur intersection property

Recall that a Banach space has the *Mazur Intersection Property* (MIP) if every closed convex subset C of X is the intersection of closed balls of X. The following results show that (MIP) is closely related to the existence of various biorthogonal systems.

Theorem. *(See [16].) Let X be a Banach space.*

(A) *Suppose that X has a biorthogonal system $(x_i, f_i)_{i \in I}$ such that $\langle f_i \rangle_{i \in I}$ is dense in X^*. Then X admits an equivalent norm with the* (MIP).
(B) *If X is non-separable and it admits a renorming with* (MIP)*, then X has an uncountable almost-biorthogonal system* (*see Definition* 4.1).

For the convenience of the reader, we recall the notions appearing in this theorem.

Definition 4.1. A sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa}$ of pairs $(x_\alpha, f_\alpha) \in X \times X^*$ is called an *ε-biorthogonal system* $(0 \leqslant \varepsilon < 1)$ if $f_\alpha(x_\alpha) = 1$ for every $\alpha < \kappa$ and $|f_\alpha(x_\beta)| \leqslant \varepsilon$ for every $\beta \neq \alpha$. The sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa}$ is called an *almost-biorthogonal system* if it is ε-biorthogonal for some $0 \leqslant \varepsilon < 1$.

The purpose of this section is to clarify the relationship between almost-biorthogonal and biorthogonal systems occurring in the theorem of [16] stated above.

Recall (see [15, Corollary 4.11]) that if X^* is not weak*-separable then X admits an uncountable biorthogonal system, even a long Schauder basic sequence. The following result from [12, Proposition 2.2, Proposition 2.7] gives a similar condition for the existence of uncountable almost-biorthogonal systems.

Theorem. *The following are equivalent for a Banach space X:*

(a) *X has an uncountable almost-biorthogonal system.*
(b) *X^* has a w^*-non-separable equivalent dual unit ball.* □

The following is a useful piece of notation in this context. For a given Banach space X, let

$$\tau(X) := \inf\{\varepsilon > 0\colon \text{ there is an uncountable } \varepsilon\text{-biorthogonal system in } X\},$$

where $\inf\emptyset = 1$. Clearly τ is invariant under isomorphisms. Moreover, we always have $\tau(X) \leqslant 1$, while $\tau(X) < 1$ iff X has an uncountable almost-biorthogonal system. Moreover the following related notion stands naturally between the notions of biorthogonal and almost-biorthogonal systems.

Definition 4.2. Recall that a sequence $(x_\alpha)_{\alpha<\kappa}$ in a Banach space X is called ω-independent if whenever for any given subsequence $(x_{\alpha_n})_{n<\omega}$ of $(x_\alpha)_{\alpha<\kappa}$, the equation $\sum_n a_n x_{\alpha_n} = 0$ implies $a_n = 0$ for every $n < \omega$.

The simpler example of an ω-independent sequence is a biorthogonal sequence. Sersouri proved in [36] that separable Banach spaces do not have uncountable ω-independent sequences. It was proved in [12, Proposition 32, p. 108] that if $(x_\alpha)_{\alpha<\omega_1}$ is ω-independent, then for every $\varepsilon > 0$ there is an uncountable subset $\Gamma_\varepsilon \subseteq \omega_1$ and functionals $(f_\alpha^{(\varepsilon)})_{\alpha\in\Gamma_\varepsilon}$ such that $(x_\alpha, f_\alpha^{(\varepsilon)})_{\alpha\in\Gamma_\varepsilon}$ is an ε-biorthogonal sequence. This means that the existence of an ω-independent family in X implies that $\tau(X) = 0$. The following questions are now quite natural and we are going to answer them later on in this section.

Question 1. (See [12].) Is it true that if $\tau(X) < 1$, then X has an uncountable biorthogonal sequence, an ω-independent family or even $\tau(X) = 0$?

Question 2. (See [12].) Is it true that if $\tau(X) = 0$, then X has an uncountable biorthogonal sequence?

4.1. Almost-biorthogonal versus biorthogonal

We present here a generic construction to answer negatively Question 1.

Definition 4.3. Let $\varepsilon \in \mathbb{Q} \cap\,]0, 1[$. The forcing $\mathbb{P}_0 = \mathbb{P}_0(\varepsilon)$ is the following. The conditions are $p = (D_p, F_p, A_p, H_p)$ in the basic forcing such that:

(∗) For every $\gamma, \eta \in A_p$ with $\gamma < \eta$ one has that $(h_\gamma^{(p)})_\gamma = 1$ and $|(h_\gamma^{(p)})_\eta| \leqslant \varepsilon$.

The next is easy to prove.

Proposition 4.4. *The forcing $\mathbb{P}_0$ has the* (EAMP) *and* (EEP). $\square$

Let $\mathbb{G}$ be a generic filter for $\mathbb{P}_0$ and let $A_{\mathbb{G}} := \bigcup_{p\in\mathbb{G}} A_p$.

Theorem 4.5. *Let $X_{\mathbb{G}}$ be any generic space for $\mathbb{P}_0$. Then*:

(I) *The sequence $(u_\gamma, h_\gamma)_{\gamma\in A_{\mathbb{G}}}$ is ε-biorthogonal, and $\langle u_\gamma\rangle_{\gamma\in A_{\mathbb{G}}}$ is dense in $X_{\mathbb{G}}$* (*in particular $A_{\mathbb{G}}$ is uncountable*) *and in $X_{\mathbb{G},H}$.*
(II) *The spaces $X_{\mathbb{G}}$ and $X_{\mathbb{G},H}$ do not have uncountable η-biorthogonal sequences for every $\eta < \varepsilon/(1+\varepsilon)$.*

Proof. (I): It is clear from the definition that $(u_\gamma, h_\gamma)_\gamma$ is ε-biorthogonal. It follows from the fact that $\mathbb{P}_0$ has (EEP) that $(u_\gamma)_{\gamma\in A_{\mathbb{G}}\cap\alpha}$ is dense in $X_{\mathbb{G}}^{(\alpha)}$ and in $X_{\mathbb{G},H}^{(\alpha)}$ (Proposition 3.20).

(II): Since $X_{\mathbb{G},H}$ is a quotient of $X_{\mathbb{G}}$ it suffices to prove the desired result for $X_{\mathbb{G}}$. We argue by contradiction. Suppose that $(y_\alpha, g_\alpha)_{\alpha<\omega_1}$ is an η-biorthogonal for some $0 \leqslant \eta < \varepsilon/(1+\varepsilon)$.

Claim 4.5.1. *There is some uncountable $\Gamma \subseteq \omega_1$ and $\delta > 0$ such that for every $0 < m, n \in \mathbb{N}$ with $m/(2n) = \varepsilon$ and every $\alpha_0 < \cdots < \alpha_{2n+1}$ in Γ one has that*

$$\left\|(y_{\alpha_0} - y_{\alpha_1}) - \frac{1}{m}\sum_{i=1}^{n}(y_{\alpha_{2i}} - y_{\alpha_{2i+1}})\right\| \geqslant \delta. \tag{37}$$

Proof. Let $\Gamma \subseteq \omega_1$ be uncountable such that

$$a := \sup_{\gamma\in\Gamma} \|g_\gamma\| < \infty.$$

Now if $\alpha_0 < \cdots < \alpha_n$ are in Γ, and $0 < m, n \in \mathbb{N}$ are such that $m/(2n) = \varepsilon$, then

$$\begin{aligned}\left\|(y_{\alpha_0} - y_{\alpha_1}) - \frac{1}{m}\sum_{i=1}^{n}(y_{\alpha_{2i}} - y_{\alpha_{2i+1}})\right\| &\geqslant \left|\frac{g_{\alpha_0}}{a}\left((y_{\alpha_0} - y_{\alpha_1}) - \frac{1}{m}\sum_{i=1}^{n}(y_{\alpha_{2i}} - y_{\alpha_{2i+1}})\right)\right| \\ &\geqslant \frac{1}{a}\left(1 - \eta - \frac{\eta\cdot 2n}{m}\right) = \frac{1}{a}\left(1 - \frac{\eta(1+\varepsilon)}{\varepsilon}\right) \\ &= \delta > 0. \qquad \square\end{aligned}$$

We fix such $\delta > 0$ and $\Gamma \subseteq \omega_1$ uncountable. Let $n, m \in \mathbb{N}$ be such that $m > 2/\delta$ and $m/2n = \varepsilon$. Let $\mathfrak{P}(v_0, \dots, v_{2n-1})$ be the following metric configuration

$$\left\|(v_0 - v_1) - \frac{1}{m}\sum_{i=1}^{n}(v_{2i} - v_{2i+1})\right\| < \frac{\delta}{2}. \tag{38}$$

Claim 4.5.2. *The configuration $\mathfrak{P}(v_0, \dots, v_{2n+1})$ is unavoidable for $\mathbb{P}_0$.*

Assuming this claim holds, let $(x_\alpha)_{\alpha\in\Gamma}$ be a sequence of points of $c_{00}(\omega_1,\mathbb{Q})$ such that

$$\|x_\alpha - y_\alpha\| \leqslant \frac{\delta}{4(n+1)}, \tag{39}$$

for every $\alpha\in\Gamma$. Then by Theorem 3.36 there are $\alpha_0<\cdots<\alpha_{2n-1}$ such that $\mathfrak{P}(x_{\alpha_0},\dots,x_{\alpha_{2n-1}})$ holds in $X_{\mathbb{G}}$. This clearly contradicts (37). It remains to prove Claim 4.5.2.

Proof of Claim 4.5.2. Let $(p_i,v_i)_{i<2n+2}$ be any Δ-system with type $t=(N,F,A,H,v)$ and root R. Let also $\theta_i:N\to D_i$ be the corresponding order-preserving bijections. We define $p=(\bigcup_{i<2n+2}D_i,F_p,\bigcup_{i<2n+2}A_i,H_p)$ where F_p is the minimal symmetric subset of $c_{00}(D_p,\mathbb{Q})$ such that:

(a) It contains $\bigvee_{i<2n+2}\theta_i(g)$ for every type $g\in F$.
(b) For every $j=0,1$ and $\gamma\in D_j\setminus R$, if $g_j\in F$ denotes the type of $h_\gamma^{(j)}$ then

$$h_\gamma^{(j)} \vee \bigvee_{i=1}^{n}(-1)^j\varepsilon\cdot\theta_{2i+j}(g_j)\in F_p.$$

(c) For every $1<j\leqslant 2n+1$ and $\gamma\in D_j\setminus R$ one has that $h_\gamma^{(j)}\in F_p$.

Now we pass to declare H_p, so we fix $\gamma\in D_p$. Now let $i<2n+2$ be such that $\gamma\in D_i$ and let g be the type of $h_\gamma^{(i)}$. Then

$$h_\gamma^{(p)}=\begin{cases}\bigvee_{i\leqslant 2n+1}\theta_i(g) & \text{if } \gamma\in R,\\ h_\gamma^{(i)} & \text{if } \gamma\in D_i\setminus R \text{ for some (unique) } 2\leqslant i\leqslant 2n+1,\\ h_\gamma^{(i)}\vee\bigvee_{j=1}^{n}(-1)^i\varepsilon\cdot\theta_{2j+i}(g) & \text{if } i=0,1 \text{ and } \gamma\notin R.\end{cases}$$

It is routine to prove that $p\in\mathbb{P}_2$ and that $p\leqslant p_i$ for every $i\leqslant 2n+1$. We prove that $\|w\|_p<\delta/2$, where

$$w=(v_0-v_1)-\frac{1}{m}\sum_{i=1}^{n}(v_{2i}-v_{2i+1}).$$

Let $f\in F_p$. If f is as in (a), then clearly $f(w)=0$. Suppose now that f is as in (b), and suppose that $f=h_\gamma^{(0)}\vee\bigvee_{i=1}^{n}(\varepsilon\cdot\theta_{2i}(g))$, where g is the corresponding type of $h_\gamma^{(0)}$. Then it follows that

$$f(w)=g(v)-\frac{n}{m}\big(\varepsilon g(v)\big)=g(v)\left(1-\frac{n}{m}\varepsilon\right)=0.$$

The case when $f=h_\gamma^{(1)}\vee\bigvee_{i=1}^{n}(-\varepsilon\cdot\theta_{2i+1}(g))$ also gives that $f(w)=0$. Finally, if f is as in (c), then $|f(w)|\leqslant 1/m<\delta/2$. $\square$

Corollary 4.6. *The space $X_{\mathbb{G}}$ has uncountable almost-biorthogonal system but it does not have uncountable ω-independent sequences.* $\square$

Remark 4.7. In a similar way one can show that the generic space $X_{\mathbb{G}}$ of $\mathbb{P}_\varepsilon$ does not contain, for every $\eta < \varepsilon$, an uncountable η-biorthogonal sequences $(x_\alpha, f_\alpha)_{\alpha<\omega_1}$ with the property that $f_\beta(x_\alpha) = 0$ for every $\alpha < \beta$.

4.2. *ε-biorthogonal for every $\varepsilon > 0$ but no biorthogonal systems*

In this subsection we give an answer to Question 2. More precisely, we prove that there is a generic space $X_{\mathbb{G}}$ such that $\tau(X_{\mathbb{G}}) = 0$ (i.e., having uncountable ε-biorthogonal sequences for every $\varepsilon > 0$), but having no uncountable biorthogonal sequences. In fact, we produce a generic space with a fundamental sequence $(u_\gamma)_{\gamma\in A}$ such that for every $\varepsilon > 0$ there is a sequence of functionals $(f_\gamma^\varepsilon)_\gamma$ such that $(u_\gamma, f_\gamma^{(\varepsilon)})_{\gamma\in A}$ is an ε-biorthogonal sequence.

Definition 4.8. Let $\mathbb{P}_1$ be the following partial ordering: The conditions are $p = (D_p, F_p, A_p, H_p)$ where:

(1) For every $\gamma \in A_p$ one has that $(h_\gamma^{(p)})_\gamma \in \{1/n\}_{n\geqslant 1}$. Set $A_p^{(n)} := \{\gamma \in A_p\colon (h_\gamma^{(p)})_\gamma = 1/n\}$.
(2) For every $\gamma, \eta \in A_p^{(n)}$ with $\gamma < \eta$ one has that

$$\left|\left(h_\gamma^{(p)}\right)_\eta\right| \leqslant \frac{1}{n^2}.$$

Proposition 4.9. *$\mathbb{P}_1$ has both* (EAMP) *and* (EEP).

Proof. It is easy to see that $\mathbb{P}_1$ has (EAMP) and (EP). To check (EEP), use $\Delta_{\mathbb{P}_1} = \{1/n\}_{n\geqslant 1}$. $\square$

Let $\mathbb{G}$ be a generic filter of $\mathbb{P}_1$. It follows from Proposition 4.9 that $X_{\mathbb{G}}$ is a Gurarij space whose dual unit ball is w^*-separable.

For every $n \in I_{\mathbb{G}}$, define $A_{\mathbb{G}}^{(n)} = \bigcup_{p\in\mathbb{G}} A_p^{(n)}$. For each $(n, \gamma) \in I_{\mathbb{G}} \times A_{\mathbb{G}}^{(n)}$, we write $h_{(n,\gamma)}$ to denote $h_{(n,u_\gamma)}^{\mathbb{G}}$. Recall that a sequence $(x_\alpha, f_\alpha)_\alpha \in X \times X^*$ is called a semi-biorthogonal system if $f_\alpha(x_\alpha) = 1$, and for every $\alpha < \beta$ one has that $f_\beta(x_\alpha) = 0$ while $f_\alpha(x_\beta) \geqslant 0$. It is clear that a biorthogonal system is a semi-biorthogonal system. We will have a more complete discussion about semi-biorthogonal sequences in Section 7.1.

Theorem 4.10.

(I) *For each $n \in \mathbb{N}^*$ the sequence $(u_\gamma, n\cdot h_\gamma)_{\gamma\in A_{\mathbb{G}}^{(n)}}$ is fundamental and $(1/n)$-biorthogonal.*
(II) *The generic space $X_{\mathbb{G}}$ does not have uncountable semi-biorthogonal sequences.*

Proof. (I): It is easy to see that for every $n \in \mathbb{N}^*$ the sequence $(u_\gamma, n\cdot h_{n,\gamma})_{\gamma<\omega_1}$ is a $1/n$-biorthogonal sequence. It follows from Proposition 3.20 that $(u_\gamma)_{\gamma\in A_{\mathbb{G}}^{(n)}\cap\alpha}$ is dense in $X_{\mathbb{G}}^{(\alpha)}$ for every limit ordinal $\alpha < \omega_1$.

The second part (II) of the statement follows from the fact that the forcing notion $\mathbb{P}_1$ has a stronger property than the (AMP), called the strong amalgamation property, that will be treated in detail in Section 7.1. $\square$

We finish this section by weakening Question 2 as follows.

Problem 1. (See [12].) Is it true that if a Banach space has an ω-independent family then it has an uncountable biorthogonal sequence?

4.3. Polyhedral spaces

Recall the following well-known notions.

Definition 4.11. Let X be a Banach space.

(A) X is called *polyhedral* if for every f.d. subspace F of X the unit ball B_F of F has finitely many extremal points.
(B) The norm of X *depends on finitely many coordinates* when for every $x \in X$, $x \neq 0$, there is $\varepsilon > 0$ and finitely many $f_0, \dots, f_n \in S_{X^*}$ such that whenever $y, z \in X$ are such that $\|y - x\|$, $\|z - x\| < \varepsilon$ and $f_i(y) = f_i(z)$ for every $i \leqslant n$, then $\|y\| = \|z\|$.

The next result relates the two notions in the class of separable spaces.

Theorem. *(See [8].) Let X be a separable Banach space. Then the following conditions are equivalent.*

(A) *There is an equivalent norm $\|\cdot\|'$ on X such that $(X, \|\cdot\|')$ is polyhedral.*
(B) *There is an equivalent norm $\|\cdot\|'$ on X which depends on finitely many coordinates.*

Recall also that separable preduals of ℓ_1 has an equivalent polyhedral norm (see [6]). However the corresponding results for arbitrary Banach spaces seem open. In particular, we do not know if any of the Asplund spaces we constructed in this section admit an equivalent polyhedral norm. But we now present a variation of these constructions whose norm is both polyhedral and depends on finitely many coordinates. Moreover, the corresponding space fails the (MIP) and it does not have uncountable biorthogonal systems. In connection with this we mention the following problem from [41].

Problem 2. Suppose that the norm of X depends on finitely coordinates. Does X admit an equivalent C^∞-smooth norm?

Definition 4.12. For $\varepsilon < 1$, define the partial ordering $\mathbb{P}$ as the set of conditions $p = (D_p, H_p)$ such that $D_p \subseteq \omega_1$ is finite, and $H_p = \{h_\gamma^{(p)}\}_{\gamma \in D_p} \subseteq c_{00}(D_p, \mathbb{Q} \cap [-1, 1])$ is such that for every $\gamma \in D_p$, $(h_\gamma^{(p)})_\gamma = 1$ and $(h_\gamma^{(p)}) \restriction \gamma = 0$.

We define the extension $p \leqslant q$ by $D_q \subseteq D_p$, $H_q \subseteq H_p \restriction D_q$

$$h_\gamma^{(p)} \restriction D_q \in \delta \cdot \mathrm{conv}_{\mathbb{Q}}(\pm H_q) \quad \text{for every } \gamma \in D_p \setminus D_q.$$

Note that $p \leqslant q$ if and only if $D_q \subseteq D_p$, $H_q \subseteq H_p \restriction D_q$ and

$$h_\gamma^{(p)} \restriction D_q \in \varepsilon \cdot \mathrm{conv}_{\mathbb{Q}}(F_q) \quad \text{for every } \gamma \in D_p \setminus D_q.$$

Proposition 4.13. *$\mathbb{P}$ has the Shanin property and $D_{\mathbb{G}} = \omega_1$ for every generic filter $\mathbb{G}$.*

Proof. If $(p_i)_{i<k}$ is a Δ-system in $\mathbb{P}$, then its basic amalgamation $p = (\bigcup_{i<k} D_i, H_p)$ is defined by $h_\gamma^{(p)} = \bigcup_{i<k} h_\gamma^{(p_i)}$ for γ in the root R of the Δ-system, and $h_\gamma^{(p)} = h_\gamma^{(p_i)}$ for $\gamma \in D_i \setminus R$. It is clear that $p \in \mathbb{P}$. We check that $p \leqslant p_i$ for every $i < k$: Fix $i < k$ and let $\gamma \in D_p \setminus D_i$. Then $\gamma \notin R$ and so $\gamma \in D_j$ for some $j \neq i$. By definition, $h_\gamma^{(p)} = h_\gamma^{(p_j)}$ and consequently, $h_\gamma^{(p)} \upharpoonright D_i = h_\gamma^{(p_j)} \upharpoonright D_i = 0$.

The proof of the fact that $D_{\mathbb{G}} = \omega_1$ follows from the fact that for every $\gamma < \omega_1$ the set $\mathcal{D}_\gamma$ of conditions p such that $\gamma \in D_p$ is dense: If $\gamma \notin D_p$, then $q = (D_p \cup \{\gamma\}, H_p \cup \{h_\gamma^{(q)} := u_\gamma\})$ is in $\mathbb{P}$ and $q \leqslant p$, because $h_\gamma^{(q)} \upharpoonright D_q = 0$. $\square$

We easier the notation by writing h_γ and X_γ to denote $h_\gamma^{(\mathbb{G})}$ and $X_{\mathbb{G}}^{(\gamma)}$ for a generic filter $\mathbb{G}$ and $\gamma < \omega_1$.

Theorem 4.14. *Let $\mathbb{G}$ be a generic filter of $\mathbb{P}$.*

(1) *Let α be a limit ordinal. Then $(h_\gamma \upharpoonright X_\alpha)_{\gamma<\omega_1}$ is a basis of $(X_\alpha)^*$ which is 1-equivalent to the unit basis of $\ell_1(\omega_1)$. Consequently, $X_{\mathbb{G}}$ is Asplund.*
(2) *$X_{\mathbb{G}}$ is polyhedral.*
(3) *The norm $\|\cdot\|_{\mathbb{G}}$ depends only on finitely many coordinates.*
(4) *The space $X_{\mathbb{G}}$ does not have uncountable biorthogonal sequences.*
(5) *The space $X_{\mathbb{G}}$ does not have the Mazur intersection property.*

Proof. (1) is proved by a density argument similarly as for the Asplund spaces $X_{\mathbb{G},H}$ above.

(2): Let us prove that $X_{\mathbb{G}}$ is polyhedral.

Claim 4.14.1. *Fix a non-zero $x \in X_{\mathbb{G}}$, and fix $\delta > 0$ such that*

$$\delta + \varepsilon(1+\delta) < 1. \tag{40}$$

If $p \in \mathbb{G}$ and $y \in X_p$ are such that $\|x-y\| < \delta\|x\|$, then

$$\|x\| = \max_{\gamma \in D_p} \left|h_\gamma(x)\right|. \tag{41}$$

Proof. Let x, y and $p \in \mathbb{G}$ be as in the hypothesis. It follows from the definition of the forcing extension $\leqslant$ that for every $\gamma \notin D_p$ one has that $\|h_\gamma \upharpoonright D_p\|_{(X_p)^*} \leqslant \varepsilon$, so

$$\left|h_\gamma(y)\right| \leqslant \varepsilon\|y\|_{X_p} = \varepsilon\|x\|.$$

Hence

$$\begin{aligned}\left|h_\gamma(x)\right| &\leqslant \|x-y\| + \left|h_\gamma(y)\right| \leqslant \varepsilon\|x\| < \delta\|x\| + \varepsilon\|y\| \leqslant \delta\|x\| + \varepsilon\big(\|x\|\big) \\ &\leqslant \big(\delta + \varepsilon(1+\delta)\big)\|x\| < \|x\|. \qquad \square\end{aligned}$$

Suppose that $F \subseteq X_{\mathbb{G}}$ is f.d. and let $(x_i)_{i<n}$ be a normalized basis of F. In addition we require that $(x_i)_{i<n}$ be an Auerbach basis. Let $\delta > 0$ satisfy (40), and let $(y_i)_{i<n}$ be in $c_{00}(\omega_1)$ such that $\|x_i - y_i\|_{\mathbb{G}} < \delta/n$. Notice that if $x = \sum_{i<n} a_i x_i$ is normalized, then

$$\left\| \sum_{i<n} a_i x_i - \sum_{i<n} a_i y_i \right\| \leqslant \max_{i<n} |a_i| \sum_{i<n} \|x_i - y_i\| < \delta. \tag{42}$$

Let $p \in \mathbb{G}$ be such that $\{x_i\}_{i<n} \subseteq X_p$. By the claim and by (42), if $x = \sum_{i<n} a_i x_i$ is normalized, then

$$1 = \|x\| = \max_{\gamma \in D_p} \left| h_\gamma(x) \right|.$$

This means, that $\{h_\gamma \restriction F\}_{\gamma \in D_p}$ is a boundary of F, hence $\mathrm{Ext}(B_{F^*}) \subseteq \{\pm h_\gamma\}_\gamma$.

The proof of (3) is very similar to the proof of (2), so we leave it to the interested reader.

To prove (4) we follow the arguments from the proof of Theorem 4.5. Finally to prove (5), notice that the extremal points of $B_{X_{\mathbb{G}}^*}$ are $\{h_\gamma\}_{\gamma<\omega_1}$, which are clearly not norm dense in $S_{X_{\mathbb{G}}}$. □

5. Lindelöf property for the weak topology

The intention of this section is to study the hereditary Lindelöf property of generic spaces equipped with their weak topologies. Recall that the weak topology of a Banach space X is the topology on X such that given $x \in X$, $f_0, \dots, f_n$ in X^* and $\varepsilon > 0$ the sets

$$U(x, f_0, \dots, f_n, \varepsilon) := \left\{ y \in X: \max_{i \leqslant n} \left| f_i(x) - f_i(y) \right| < \varepsilon \right\}$$

form an open neighborhood basis on x. Recall that a topological space T is called Lindelöf if every open cover of T has a countable subcover. The space T is called hereditarily Lindelöf (HL in short) if every subspace is Lindelöf. A sequence $(x_\alpha)_{\alpha<\kappa}$ in a topological space T is called *right separated* if for every $\alpha < \kappa$ one has that x_α is not in the closure of $\{x_\beta: \alpha < \beta < \kappa\}$. It is a simple exercise to prove that T is HL if and only if T does not have uncountable right separated sequences. A weak-right separated sequence in a Banach space X is a right separated sequence in X endowed with its weak topology. It is clear that if $(x_\alpha, f_\alpha)_\alpha$ is an almost-biorthogonal sequence, then $(x_\alpha)_\alpha$ is a weak-right separated sequence. In this section, we shall examine the following two natural questions.

Question 3. Is it true that if a Banach space X does not have an uncountable almost-biorthogonal system then X is hereditarily Lindelöf relative to its weak topology?

Question 4. Is it true that if X is hereditary Lindelöf relative to its weak topology, then so is its square X^2?

5.1. Weak topology right-separated sequences and almost-biorthogonal systems

We describe a generic Banach space $X_{\mathbb{G}}$ having an uncountable weak right-separated sequence but without almost-biorthogonal systems answering thus Question 3.

Definition 5.1. Let $I =]1/2, 1]$ and $J =]-1/2, 1/2[$. Let $\mathbb{P}_2$ be the forcing notion consisting on all the basic conditions $p = (D_p, F_p, A_p, H_p) \in \mathbb{P}_{\text{basic}}$ such that:

(1) There is $B_p \subseteq A_p$ such that $B_p \cap (B_p + 1) = \emptyset$ such that $A_p = B_p \cup (B_p + 1)$. For every $\gamma \in B_p$ write $f_\gamma^{(p)}$ and $g_\gamma^{(p)}$ to denote $h_\gamma^{(p)}$ and $h_{\gamma+1}^{(p)}$, respectively.
(2) For every $\gamma \in B_p$ one has that $(f_\gamma^{(p)})_\gamma = 1$ and $(g_\gamma^{(p)})_{\gamma+1} = 3/4$.
(3) For every $\gamma, \eta \in B_p$ with $\gamma < \eta$ one has that

$$\left(\left(f_\gamma^{(p)}\right)_\eta, \left(g_\gamma^{(p)}\right)_\eta\right) \notin I \times J. \tag{43}$$

Given a generic filter $\mathbb{G}$ for $\mathbb{P}_2$, and $\gamma \in B_{\mathbb{G}} := \bigcup_{p \in \mathbb{G}} B_p$ we write f_γ and g_γ to denote $f_\gamma^{\mathbb{G}}$ and $g_\gamma^{\mathbb{G}}$ respectively.

Proposition 5.2. *The forcing $\mathbb{P}_2$ has the* (EAMP) *and the* (EEP).

Proof. Clearly $\mathbb{P}_2$ has the basic amalgamation property and it is hereditary. It is spreading because the spreading property only mentions order-preserving mappings that respect the successor operation. So if $p \in \mathbb{P}_3$, and $\theta : D_p \to \omega_1$ is one of such mappings, then $\theta(s) = (\theta(\gamma), \theta(\gamma) + 1)$ for every $s \in A_p$ and hence $\theta(p) \in \mathbb{P}_3$. The rest of the requirements in (EP) is easy to verity. To check (EEP), use $\Delta_{\mathbb{P}_2} = \{1, 3/4\}$. We leave the rest of the details to the reader. $\square$

Let $\mathbb{G}$ be a generic filter for $\mathbb{P}_2$.

Theorem 5.3. *The sequence $(u_\gamma)_{\alpha \in B_{\mathbb{G}}}$ is a fundamental weak right separated sequence of both $X_{\mathbb{G}}$ and $X_{\mathbb{G},H}$, but neither $X_{\mathbb{G}}$ nor $X_{\mathbb{G},H}$ have uncountable almost-biorthogonal systems.*

Proof. Since $B_{\mathbb{G}} = A_{\mathbb{G}}^{(1)}$, it follows from Proposition 3.20 that for every α limit $\langle u_\gamma \rangle_{\gamma \in B_{\mathbb{G}} \cap \alpha}$ is dense in $(X_{\mathbb{G}})^{(\alpha)}$ and in $(X_{\mathbb{G},H})^{(\alpha)}$. The sequence $(u_\gamma)_{\gamma \in B_{\mathbb{G}}}$ is clearly right separated family with respect to the weak topology of $X_{\mathbb{G},H}$, because by definition for every $\gamma < \eta$ in $B_{\mathbb{G}}$ one has that

$$u_\eta \notin U\left(u_\gamma, f_\gamma, g_\gamma, \frac{1}{2}\right).$$

Now we prove that $X_{\mathbb{G}}$ does not have uncountable almost-biorthogonal sequences. It is easy to see that if $(y_\alpha)_{\alpha < \omega_1}$ is an uncountable normalized almost-biorthogonal sequence, then there is $\varepsilon > 0$ and an uncountable subsequence $(y_\alpha)_{\alpha \in \Gamma}$ such that for every $n \in \mathbb{N}$ and every $\alpha_0 < \cdots < \alpha_n$ in Γ one has that

$$\left\| y_{\alpha_0} - \frac{1}{n} \sum_{i=1}^{n} y_{\alpha_i} \right\| > \varepsilon. \tag{44}$$

So, a simple approximation argument gives that in order to prove that $X_{\mathbb{G}}$ does not have uncountable almost-biorthogonal sequences it suffices to prove the following.

Claim 5.3.1. *The metric configuration* $\mathfrak{P}(v_0, \dots, v_n)$ *defined by*

$$\left\| v_0 - \frac{1}{n} \sum_{i=1}^{n} v_i \right\| \leqslant \frac{1}{n} \max_{i \leqslant n} \|v_i\| \tag{45}$$

is unavoidable for $\mathbb{P}_2$.

Proof. We use Theorem 3.36. Let $(p_i, v_i)_{i \leqslant n}$ be a Δ-sequence of type $t = (N, F, A, H, v)$ and root R. For each $i \leqslant n$, let $\theta_i : N \to D_i$ be the corresponding order-preserving bijections. Let $A \subseteq N$ be the type of B_i, and for each $k \in B$, let $f_k, g_k \in F$ be the types of $f^{p_0}_{\theta_0(k)}$ and $g^{p_0}_{\theta_0(k)}$ respectively. For each $k < N$ let

$$c_k = e_k(v) \quad \text{and} \quad d_k = h_k(v).$$

Let $p = (\bigcup_{i \leqslant n} D_i, F_p, \bigcup_{i \leqslant n} A_i, H_p) \in \mathbb{P}_2$ be defined as follows. $B_p = \bigcup_{i \leqslant n} B_i$ and F_p is the minimal symmetric subset of $c_{00}(\bigcup_{i \leqslant n} D_i, \mathbb{Q})$ such that:

(a) It contains $\bigvee_{i \leqslant n} \theta_i(h)$ for every type $h \in F$. In particular, if $\gamma \in B_p \cap R$, then we set

$$f_\gamma^{(p)} := \bigvee_{i \leqslant n} f_\gamma^{(p_i)}, \qquad g_\gamma^{(p)} := \bigvee_{i \leqslant n} g_\gamma^{(p_i)}.$$

(b) Let $\gamma \in \bigcup_{i \leqslant n} D_i \setminus R$.

(b.0) Suppose that $\gamma \in B_i \setminus R$ with $0 < i \leqslant n$. Then F_p contains $f_\gamma^{(p_i)}$ and $g_\gamma^{(p_i)}$, and we set

$$f_\gamma^{(p)} := f_\gamma^{(p_i)}, \qquad g_\gamma^{(p)} := g_\gamma^{(p_i)}.$$

(b.1) Suppose that $\gamma \in B_0$. Let $k = \theta_0^{-1}(\gamma)$.

(b.1.0) Suppose first that $|c_k| \leqslant |d_k|$. Let $m \leqslant n$ be such that

$$\frac{m}{n} \leqslant \frac{|c_k|}{|d_k|} < \frac{m+1}{n}. \tag{46}$$

Let τ be the sign of c_k/d_k. Then F_p contains $\theta_0(f_k) \vee \bigvee_{i=1}^{m} \tau \cdot \theta_i(g_k)$ and we set

$$f_\gamma^{(p)} = \theta_0(f_k) \vee \bigvee_{i=1}^{m} \tau \cdot \theta_i(g_k), \qquad g_\gamma^{(r)} = \bigvee_{i \leqslant n} \theta_i(g_k).$$

(b.1.1) Suppose now that $|c_k| > |d_k|$. Let $m \leqslant n$ be such that

$$\frac{m}{n} \leqslant \frac{|d_k|}{|c_k|} < \frac{m+1}{n}.$$

Let τ be the sign of c_k/d_k. Then F_p contains $\theta_0(g_k) \vee \bigvee_{i=1}^{m} \tau \cdot \theta_i(f_k)$ and we set

$$f_\gamma^{(p)} = \bigvee_{i \leqslant n} \theta_i(f_k), \qquad g_\gamma^{(p)} = \theta_0(g_k) \vee \bigvee_{i=1}^{m} \tau \cdot \theta_i(f_k).$$

(c) Let $\gamma \in \bigcup_{i \leqslant n} D_i \setminus B_p$. If $\gamma \in R$, then we declare

$$h_\gamma^{(p)} := \bigvee_{i \leqslant n} h_\gamma^{(p_i)}.$$

If $\gamma \in D_i \setminus R$, and if g is the type of $h_\gamma^{(p_i)}$, then we declare

$$h_\gamma^{(p)} := \begin{cases} \bigvee_{j<n} \theta_j(g) & \text{if } i = 0, \\ h_\gamma^{(p_i)} & \text{if } i > 0. \end{cases}$$

We check that $p \in \mathbb{P}_2$, and that $p \leqslant p_i$ for every $i \leqslant n$. The only nontrivial part is to prove that the declared $f_\gamma^{(p)}$ and $g_\gamma^{(p)}$, $\gamma \in B_p = \bigcup_{i \leqslant n} B_i$, have the required properties. So, we fix $\gamma \in B_i$ for some $i \leqslant n$. First of all, observe that $f_\gamma^{(p)} \restriction D_i = f_\gamma^{(p_i)}$ and $g_\gamma^{(p)} \restriction D_i = g_\gamma^{(p_i)}$, it follows that $(f_\gamma^{(p)})_\gamma = 1$ and $(g_\gamma^{(p)})_{\gamma+1} = 3/4$. Now, let $\eta \in A_j$ with $j \leqslant n$ and $\gamma < \eta$.

CASE 1. Suppose first that $\gamma \in R$. Then $f_\gamma^{(p)} = \bigvee_{i \leqslant n} f_\gamma^{(p_i)}$ and $g_\gamma^{(p)} = \bigvee_{i \leqslant n} g^{(p_i)}$. Then,

$$\bigl(\bigl(f_\gamma^{(p)}\bigr)_\eta, \bigl(g_\gamma^{(p)}\bigr)_\eta\bigr) = \bigl(\bigl(f_\gamma^{(p_j)}\bigr)_\eta, \bigl(g_\gamma^{(p_j)}\bigr)_\eta\bigr) \notin I \times J. \tag{47}$$

CASE 2. Suppose now that $\gamma \notin R$ and $0 < i \leqslant n$. Then $f_\gamma^{(p)} = f_\gamma^{(p_i)}$ and $g_\gamma^{(p)} = g_\gamma^{(p_i)}$. Now if $i < j \leqslant n$, then clearly

$$\bigl(\bigl(f_\gamma^{(p)}\bigr)_\eta, \bigl(g_\gamma^{(p)}\bigr)_\eta\bigr) = (0, 0) \notin I \times J. \tag{48}$$

While if $j = i$, then it follows that

$$\bigl(\bigl(f_\gamma^{(p)}\bigr)_\eta, \bigl(g_\gamma^{(p)}\bigr)_\eta\bigr) = \bigl(\bigl(f_\gamma^{(p_i)}\bigr)_\eta, \bigl(g_\gamma^{(p_i)}\bigr)_\eta\bigr) \notin I \times J. \tag{49}$$

CASE 3. Suppose that $\gamma \notin R$ and $i = 0$. If $j = 0$, then

$$\bigl(\bigl(f_\gamma^{(p)}\bigr)_\eta, \bigl(g_\gamma^{(p)}\bigr)_\eta\bigr) = \bigl(\bigl(f_\gamma^{(p_0)}\bigr)_\eta, \bigl(g_\gamma^{(p_0)}\bigr)_\eta\bigr) \notin I \times J. \tag{50}$$

Suppose that $j > 0$. If $m < j \leqslant n$, then

$$\bigl(\bigl(f_\gamma^{(p)}\bigr)_\eta, \bigl(g_\gamma^{(p)}\bigr)_\eta\bigr) = (0, 0) \notin I \times J. \tag{51}$$

Now suppose that $j \leqslant m$. Suppose first that $c_k \leqslant d_k$. Let $k = \theta_0^{-1}(\gamma)$. Then if $(g_\gamma^{(p_j)})_\eta \notin J$, then

$$\bigl(\bigl(f_\gamma^{(p)}\bigr)_\eta, \bigl(g_\gamma^{(p)}\bigr)_\eta\bigr) = \bigl(\tau \cdot \bigl(g_\gamma^{(p_j)}\bigr)_\eta, \bigl(g_\gamma^{(p_j)}\bigr)_\eta\bigr) \notin I \times J. \tag{52}$$

Otherwise, $(g_\gamma^{(p_i)})_\eta \in J =]-1/2, 1/2[$, and, using that $I \cap J = I \cap -J = \emptyset$, it follows that

$$\big(\big(f_\gamma^{(p)}\big)_\eta, \big(g_\gamma^{(p)}\big)_\eta\big) = \big(\tau\cdot\big(g_\gamma^{(p_j)}\big)_\eta, \big(g_\gamma^{(p_j)}\big)_\eta\big) \notin I\times J. \tag{53}$$

The case when $d_k < c_k$ is proved in the same way. We leave the details to the reader.

It remains now to prove that

$$\left\| v_0 - \frac{1}{n}\sum_{i=1}^n v_i \right\|_p \leqslant \frac{\max_{i\leqslant n}\|v_i\|_p}{n} = \frac{\|v\|_t}{n}. \tag{54}$$

We set $w = v_0 - (1/n)\sum_{i=1}^n v_i$. Let $f \in F_p$. It is clear that if f is as in (a), then $f(w) = 0$, and if f is as in (b.0), then $|f(w)| \leqslant \|v\|_p/n$. Now suppose that f is as in the case (b.1.0), i.e.,

$$f = \theta_0(f_k) \vee \bigvee_{i=1}^m \tau\cdot\theta_i(g_k)$$

where $|R| \leqslant k < N$, $|c_k| \leqslant |d_k| \leqslant \|v\|_t$, τ is the sign of c_k/d_k and $m \leqslant n$ is such that (46) holds. It follows that

$$\big|f(z)\big| = \left|e_k(x) - \frac{1}{n}\sum_{i=1}^m h_k(x)\right| = \left|c_k - \frac{m\cdot\tau}{n}d_k\right| = |c_k| - \frac{m}{n}|d_k|, \tag{55}$$

and hence, by (46),

$$\big|f(z)\big| = |c_k| - \frac{m}{n}|d_k| = |c_k| - \frac{m+1}{n}|d_k| + \frac{|d_k|}{n} \leqslant \frac{|d_k|}{n} \leqslant \frac{\|v\|_t}{n}. \tag{56}$$

Finally, in a similar way than for the case (b.1.0) one proves that if f is as in (b.1.1) then $|f(z)| \leqslant 1/n$. We leave the details to the interested reader. □

5.2. HL in finite powers

We concentrate now on Question 4. We provide a generic Banach space $X_{\mathbb{G}}$ such that $(X, w)^n$ is hereditarily Lindelöf but $(X, w)^{n+1}$ is not. We present only the case $n = 1$ leaving the details of the general case to the interested reader.

Definition 5.4. Let $\mathbb{P}_3$ be the forcing notion consisting on all basic conditions $p \in \mathbb{P}_{\text{basic}}$, $p = (D_p, F_p, A_p, H_p)$ such that

(1) There is $B_p \subseteq A_p$ such that $B_p \cap (B_p + 1) = \emptyset$ and such that $A_p = B_p \cup (B_p + 1)$. Given $\gamma \in B_p$ we write $f_\gamma^{(p)}$ and $g_\gamma^{(p)}$ to denote $h_\gamma^{(p)}$ and $h_{\gamma+1}^{(p)}$, respectively.

(2) For every $\gamma \in B_p$ one has that

$$\big(f_\gamma^{(p)}\big)_\gamma = 1, \qquad \big(f_\gamma^{(p)}\big)_{\gamma+1} = 0 \quad\text{and}\quad \big(g_\gamma^{(p)}\big)_{\gamma+1} = 1. \tag{57}$$

(3) For every $\gamma, \eta \in B_p$ with $\gamma < \eta$ one has that

$$\left(\left(f_\gamma^{(p)}\right)_\eta, \left(g_\gamma^{(p)}\right)_\eta, \left(f_\gamma^{(p)}\right)_{\eta+1}, \left(g_\gamma^{(p)}\right)_{\eta+1}\right) \notin I \times J^2 \times I, \tag{58}$$

where $I =]1/2, 1]$ and $J =]-1/2, 1/2[$.

The next is quite easy to verify and is left to the interested reader.

Proposition 5.5. $\mathbb{P}_3$ *has* (EAMP) *and* (EEP). □

Let $\mathbb{G}$ be a generic filter for this forcing notion, and let $B_{\mathbb{G}} := \bigcup_{p \in \mathbb{G}} B_p$.

Theorem 5.6. *The generic spaces $X_{\mathbb{G}}$ and $X_{\mathbb{G},H}$ with their weak topology are hereditarily Lindelöf but their squares are not. Indeed, the sequence $(u_\gamma, u_{\gamma+1})_{\gamma \in B_{\mathbb{G}}}$ is an uncountable right separated in $(X_{\mathbb{G},H}, w)^2$ and so in $(X_{\mathbb{G}}, w)^2$.*

The proof of this result needs the following simple fact.

Proposition 5.7. *The generic space $X_{\mathbb{G}}$ is hereditary Lindelöf with its weak topology iff for every normalized sequence $(x_\alpha)_{\alpha<\omega_1}$ of points of $c_{00}(\omega_1, \mathbb{Q})$, for every $(f_i^{(\alpha)})_{(i,\alpha)\in m\times\omega_1} \in (B_{(X_{\mathbb{G}})^*})^{m\times\omega_1}$ and every $\varepsilon > 0$ there are $\alpha < \beta < \omega_1$ such that*

$$\max_{i<m} \left| f_i^{(\alpha)}(x_\alpha) - f_i^{(\alpha)}(x_\beta) \right| \leqslant \varepsilon.$$

Proof. This is done by a simple approximation argument. We leave the details to the interested reader. □

Proof of Theorem 5.6. We start by the last part of the statement. The fact that $B_{\mathbb{G}}$ is uncountable follows from the property (EEP) of $\mathbb{P}_3$. The fact that $(u_\gamma, u_{\gamma+1})_{\gamma \in B_{\mathbb{G}}}$ is right separated in $(X_{\mathbb{G},H}, w)^2$ readily follows from the definition of the forcing notion $\mathbb{P}_3$.

Now we prove that $(X_{\mathbb{G}}, w)$ is hereditarily Lindelöf, which of course gives the corresponding property of $(X_{\mathbb{G},H}, w)$. Let $0 < \varepsilon < 1$, $m \in \mathbb{N}$ and let $n \geqslant 2m/\varepsilon$. Define the configuration $\mathfrak{P}(v_0, \dots, v_n)$ by

$$\begin{gathered} \text{For every } f_0, \dots, f_{m-1} \in B_{X^*} \text{ there is } 1 \leqslant i \leqslant n \text{ with} \\ \max_{j<m} \left| f_j(v_0) - f_j(v_i) \right| \leqslant \varepsilon \max_{k \leqslant n} \|v_k\|. \end{gathered} \tag{59}$$

Claim 5.7.1. *The configuration $\mathfrak{P}(v_0, \dots, v_n)$ is unavoidable for $\mathbb{P}_3$.*

Let us see how to use the previous claim to prove that $X_{\mathbb{G}}$ is HL: We use Proposition 5.7. Fix a normalized separated sequence $(x_\alpha)_{\alpha<\omega_1}$ in $c_{00}(\omega_1, \mathbb{Q})$, $(f_j^{(\alpha)})_{(j,\alpha)\in m\times\omega_1} \in (B_{(X_{\mathbb{G}})^*})^{m\times\omega_1}$ and $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $n \geqslant 4m/\varepsilon$. Since the corresponding property $\mathcal{P}(v_0, \dots, v_n)$ is unavoidable for $\mathbb{P}_3$, Theorem 3.36 implies that there are $\alpha_0 < \cdots < \alpha_n$ such that $\mathcal{P}(x_{\alpha_0}, \ldots, x_{\alpha_n})$ holds in $X_{\mathbb{G}}$. Hence there is $1 \leqslant i \leqslant n$ such that

$$\max_{j<m} \left| f_j^{(\alpha_0)}(x_{\alpha_0}) - f_j^{(\alpha_0)}(x_{\alpha_i}) \right| \leqslant \varepsilon$$

and we are done. Let us prove Claim 5.7.1.

Proof of Claim 5.7.1. Let $(p_i, v_i)_{i \leqslant n}$ be a Δ-system in $\mathbb{P}_3$ of type $t = (N, F, A, H, v)$ and root R. Let $\theta_i : N \to D_{p_i}$ be the corresponding order-preserving bijections, for $i \leqslant n$. Let B be the type of (any) B_i. Observe that if $\gamma \in B_0 \cap R$ then $\gamma + 1 \in R$.

We define $p = (\bigcup_{i \leqslant n} D_i, F_p, \bigcup_{i \leqslant n} A_i, H_p)$, where the set F_p is the minimal symmetric subset of $c_{00}(\bigcup_{i \leqslant n} D_i, \mathbb{Q})$ with the following properties:

(a) It contains all $\bigvee_{i \leqslant n} \theta_i(g)$ for every type $g \in F$. In particular, for every $\gamma \in D_i \cap R$, and every $\eta \in D_0 \setminus A_0$ (so in particular for $\eta \in R$) we define

$$f_\gamma^{(p)} := \bigvee_{j \leqslant n} f_\gamma^{(p_j)}, \qquad g_\gamma^{(p)} := \bigvee_{j \leqslant n} g_\gamma^{(p_j)} \quad \text{and} \quad h_\eta^{(p)} := \bigvee_{j \leqslant n} h_\eta^{(p_j)}.$$

(b) It contains all $f \in \bigcup_{i \leqslant n} F_{p_i}$ such that $f \restriction R = 0$. In particular, for every $1 \leqslant i \leqslant n$ and every $\gamma \in D_i \setminus R$ and $\eta \in D_i \setminus A_i$ we declare

$$f_\gamma^{(p)} := f_\gamma^{(p_i)}, \qquad g_\gamma^{(p)} := g_\gamma^{(p_i)} \quad \text{and} \quad h_\eta^{(p)} := h_\eta^{(p_i)}.$$

(c) Let $\gamma \in B_0 \setminus R$, let $k = \theta_0^{-1}(\gamma)$, and let

f_k be the type of $f_\gamma^{(p_0)}$, $\qquad g_k$ be the type of $g_\gamma^{(p_0)}$, $\qquad c_k = f_k(v)$, and $\quad d_k = g_k(v)$.

(c.0) Suppose that $|c_k| \leqslant |d_k|$, and $c_k \neq 0$. Then F_p contains

$$f_\gamma^{(p_0)} \vee \bigvee_{i=1}^{n} \frac{c_k}{d_k} \cdot \theta_i(g_k)$$

and we declare

$$f_\gamma^{(p)} = f_\gamma^{(p_0)} \vee \bigvee_{i=1}^{n} \frac{c_k}{d_k} \cdot \theta_i(g_k), \qquad g_\gamma^{(p)} = \bigvee_{i \leqslant n} \theta_i(g_k).$$

(c.1) Suppose that $c_k = d_k = 0$. In this case, we simply declare

$$f_\gamma^{(p)} := f_\gamma^{(p_0)}, \qquad g_\gamma^{(p)} := g_\gamma^{(p_0)}.$$

(c.2) Suppose that $|d_k| < |c_k|$. Then F_p contains

$$f_\gamma^{(p_0)} \vee \bigvee_{i=1}^{n} \frac{d_k}{c_k} \cdot \theta_i(f_k).$$

We declare

$$f_\gamma^{(p)} := \bigvee_{i \leqslant n} \theta_i(f_k), \qquad g_\gamma^{(p)} := g_\gamma^{(p_0)} \vee \bigvee_{i=1}^{n} \frac{d_k}{c_k} \cdot \theta_i(f_k).$$

Let us prove first that $p \in \mathbb{P}_3$. It is routine to prove that p is a basic condition such that $p \leqslant p_i$ for every $i \leqslant n$. We only check that p has the property (3) in Definition 5.4. So we fix $\gamma < \eta$ with $\gamma \in B_i$ and $\eta \in B_j$. In particular, $i \leqslant j$. We distinguish several cases:

CASE 1. $\gamma \in R$. Then

$$\begin{aligned} \big(\big(f_\gamma^{(p)}\big)_\eta, \big(f_\gamma^{(p)}\big)_{\eta+1}\big) &= \big(\big(f_\gamma^{(p_j)}\big)_\eta, \big(f_\gamma^{(p_j)}\big)_{\eta+1}\big), \quad \text{and} \\ \big(\big(g_\gamma^{(p)}\big)_\eta, \big(g_\gamma^{(p)}\big)_{\eta+1}\big) &= \big(\big(g_\gamma^{(p_j)}\big)_\eta, \big(g_\gamma^{(p_j)}\big)_{\eta+1}\big), \end{aligned}$$

and (58) for p holds because it is true for p_j.

CASE 2. Suppose that $\gamma \notin R$ and $1 \leqslant i \leqslant n$. If $j = i$, then we are done since the desired property holds for p_i and $f_\gamma^{(p)} \upharpoonright D_{p_i} = f_\gamma^{(p_i)}$, $g_\gamma^{(p)} \upharpoonright D_{p_i} = g_\gamma^{(p_i)}$. Otherwise $i < j$, and hence, by definition of $g_\gamma^{(p)}$, one has that $(g_\gamma^{(p)})_{\eta+1} = 0$, while $(g_\gamma^{(p)})_{\gamma+1} = 1$. This implies (58).

CASE 3. Suppose that $\gamma \notin R$ and $i = 0$. We use the same notation than in (c) above. We distinguish three subcases:

CASE 3.1. Suppose that $|c_k| \leqslant |d_k|$, and $c_k \neq 0$. Suppose that $|(\theta_j(g_k))_\eta| \geqslant 1/2$. Then it follows that

$$\big|\big(g_\gamma^{(p)}\big)_\eta\big| = \big|\big(\theta_j(g_k)\big)_\eta\big| \geqslant \frac{1}{2},$$

while $(g_\gamma^{(p)})_\eta = 0$. This implies that (58) holds. Otherwise, $|(\theta_j(g_k))_\eta| < 1/2$. This implies that

$$\big|\big(f_\gamma^{(p)}\big)_\eta\big| = \frac{|c_k|}{|d_k|} \cdot \big|\big(\theta_j(g_k)\big)_\eta\big| \leqslant \big|\big(\theta_j(g_k)\big)_\eta\big| < \frac{1}{2}$$

while $(f_\gamma^{(p)})_\gamma = 1$. This implies (58).

CASE 3.2. Suppose that $c_k = d_k$. Then one proceeds as in Case 2.

CASE 3.3. Suppose that $|d_k| < |c_k|$. This is the symmetric situation to Case 3.1. Suppose that $|(\theta_j(f_k))_{\eta+1}| \geqslant 1/2$. Then it follows that

$$\big|\big(f_\gamma^{(p)}\big)_{\eta+1}\big| = \big|\big(\theta_j(f_k)\big)_{\eta+1}\big| \geqslant \frac{1}{2},$$

while $(f_\gamma^{(p)})_{\gamma+1} = 0$. This implies that (58) holds. Otherwise, $|(\theta_j(f_k))_{\eta+1}| < 1/2$. This implies that

$$\left|\left(g_\gamma^{(p)}\right)_{\eta+1}\right| = \frac{|d_k|}{|c_k|}\left|\left(\theta_j(f_k)\right)_{\eta+1}\right| \leqslant \left|\left(\theta_j(f_k)\right)_{\eta+1}\right| < \frac{1}{2}$$

while $(g_\gamma^{(p)})_{\gamma+1} = 1$. This implies (58), and finishes the proof of the fact that $p \in \mathbb{P}_3$ and that $p \leqslant p_{\alpha_i}$ for every $i \leqslant n$.

It rests to show that the configuration $\mathfrak{P}(v_0, \dots, v_n)$ holds in X_p. Let

$$G = \left\{f \in F_p\colon\ f(v_i) = 0 \text{ for all } i \leqslant n \text{ except one } \tau = \tau(f) \leqslant n\right\}.$$

Observe that by definition of F_p, if $f \in G$ then $\tau(f) > 0$.

Claim 5.7.2. *For every $f \in F_p \setminus G$ and every $i \leqslant n$ one has that*

$$f(v_0) = f(v_i). \tag{60}$$

Proof. If f is as in (a), then the result is clear. Suppose that f is as in (c.0), i.e. $f = f_\gamma^{(p_0)} \vee \bigvee_{i=1}^n (c_k/d_k)\theta_i(g_k)$ using the terminology introduced in (c). Then $f(v_0) = f_k(v)$ and

$$f(v_i) = \frac{c_k}{d_k}\theta_i(g_i)(v_i) = \frac{c_k}{d_k} g_k(v) = f_k(v).$$

If f is as in (c.1) then $f(v_0) = 0 = f(v_i)$. Finally, if f is as in (c.2), then one proves that $f(v_0) = f(v_i)$ in a similar way that in the case (c.0). □

Fix $f_0, \dots, f_{m-1} \in B_{X_p^*}$, and for each $j < m$ let $(q_f^{(j)})_{f \in F_p}$ be a convex combination such that

$$f_j = \sum_{f \in F_p} q_f^{(j)} f.$$

For each $1 \leqslant i \leqslant n$ and $j < m$, let

$$I_{i,j} := \left\{f \in F_p\colon\ q_f^{(j)} \neq 0 \text{ and } \tau(f) = i\right\} \quad \text{and} \quad \lambda_{i,j} := \sum_{f \in I_{i,j}} q_f^{(i)}.$$

Notice that for every $j < m$, $\{I_{i,j}\}_{1 \leqslant i \leqslant n}$ is a partition of $\{f \in G\colon\ q_f^{(j)} \neq 0\}$. For each $j < m$, let

$$H_j = \left\{1 \leqslant i \leqslant n\colon\ \lambda_{i,j} > \frac{\varepsilon}{2}\right\}. \tag{61}$$

Then

$$\#\left(\bigcup_{j<m} H_j\right)\frac{\varepsilon}{2} < \sum_{j<m}\sum_{i \in H_j} \lambda_{i,j} \leqslant \sum_{j<m}\sum_{f \in F_p} q_f^{(j)} = m,$$

and, by the choice of n,

$$\#\Big(\bigcup_{j<m} H_j\Big) < \frac{2m}{\varepsilon} \leqslant n.$$

So $\bigcup_{j<m} H_j \subsetneq \{1,\dots,n\}$. Let $1 \leqslant k \leqslant n$ be such that

$$\max_{j<m} \lambda_{k,j} \leqslant \frac{\varepsilon}{2}. \tag{62}$$

We claim that k is the desired integer: Fix $j<m$. Since if $f \in F_p \setminus H_{k,j}$ then either $q_f^{(j)} = 0$ or $f \notin G$, it follows that

$$\begin{aligned}
\big|f_j(v_0) - f_j(v_k)\big| &= \Big|\sum_{f\in I_{k,j}} q_f^{(j)}\big(f(v_0)-f(v_k)\big) + \sum_{f\in F_p\setminus I_{k,j}} q_f^{(j)}\big(f(v_0)-f(v_k)\big)\Big| \\
&\leqslant \lambda_{k,j}\|v_0 - v_k\|_p \leqslant \frac{\varepsilon}{2}\cdot 2\|v\|_t = \varepsilon \cdot \max_{i\leqslant n}\|v_i\|_p,
\end{aligned}$$

as desired. □

5.3. *Support sets and the hereditary Lindelöf property*

We finish this section by providing a generic Banach space which has a support set but whose weak topology is hereditarily Lindelöf in all finite powers. To shorten the terminology, we shall sometimes say that a Banach space X is *powerfully hereditarily Lindelöf*, and if necessary to shorten this further as PHL, if for every positive integer n the product space $(X,w)^n$ is hereditary Lindelöf.

Definition 5.8. Let $\mathbb{P}_4$ be the set of all basic conditions $p=(D_p,F_p,A_p,H_p)$ such that

(∗) for every $\gamma,\eta \in A_p$ with $\gamma<\eta$ one has that

$$\big(h^{(p)}_{(1,u_\gamma)}\big)_\gamma = 1, \quad \text{and} \quad \big(h^{(p)}_{(1,u_\gamma)}\big)_\eta \geqslant 0.$$

The following fact is easy to verify.

Proposition 5.9. $\mathbb{P}_4$ *has* (EAMP) *and* (EEP). □

We fix a generic filter $\mathbb{G}$ for $\mathbb{P}_4$. Recall that $A_{\mathbb{G}} = \bigcup_{p\in\mathbb{G}} A_p$, and for $\gamma \in A_{\mathbb{G}}$, let $h_\gamma := h_\gamma^{(\mathbb{G})}$.

Theorem 5.10.

(I) *The sequence* $(u_\gamma, h_\gamma)_{\gamma\in A_{\mathbb{G}}}$ *is a fundamental semi-biorthogonal sequence of the generic spaces* $X_{\mathbb{G}}$ *and* $X_{\mathbb{G},H}$ *and*

(II) *for every* $k\in\mathbb{N}$, *the power spaces* $(X_{\mathbb{G}},w)^k$ *and* $(X_{\mathbb{G},H},w)^k$ *are hereditarily Lindelöf.*

Proof. (I): It is clear by definition that $(u_\gamma, h_\gamma)_{\gamma\in A_{\mathbb{G}}}$ is a semi-biorthogonal sequence. The fact that $(u_\gamma)_{\gamma\in A_{\mathbb{G}}}$ is fundamental (and therefore uncountable) follows from Proposition 3.20.

Let $k \in \mathbb{N}$ and let us prove that $(X_{\mathbb{G}}, w)^k$ is hereditarily Lindelöf, which gives the related result for $(X_{\mathbb{G},H}, w)^k$. Let $0 < \varepsilon < 1$, $k, m \in \mathbb{N}$ and let $n \in \mathbb{N}$ be such that $n \geqslant 2m/\varepsilon$. Let $\mathfrak{P}((v_0^{(i)})_{i<k}, \dots, (v_{n-1}^{(i)})_{i<k})$ be the following configuration:

$$\text{For every } f_0, \dots, f_{m-1} \in B_{X^*} \text{ there is } 1 \leqslant i \leqslant n \text{ such that}$$
$$\max_{l<k, j<m} \left| f_j\left(v_0^{(l)}\right) - f_j\left(v_i^{(l)}\right)\right| \leqslant \varepsilon \max_{l<k, j<n} \left\| v_j^{(l)} \right\|_X.$$

As in the proof of Theorem 5.6 it suffices to prove the following claim.

Claim 5.10.1. *The configuration $\mathfrak{P}((v_0^{(i)})_{i<k}, \dots, (v_{n-1}^{(i)})_{i<k})$ is unavoidable for $\mathbb{P}_4$.*

Proof. We use Theorem 3.36. So we fix a Δ-sequence $(p_i, (v_i^{(j)})_{j<k})_{i\leqslant n}$ of type $t = (N, F, A, H, (v^{(j)})_{j<k})$ and root R. Let also for each $i \leqslant n$, $\theta_i : N \to D_{p_i}$ be the corresponding order-preserving bijection. Let $p = (\bigcup_{i\leqslant n} D_i, F_p, \bigcup_{i\leqslant n} A_i, H_p)$ where F_p is the minimal symmetric subset of $c_{00}(\bigcup_{i\leqslant n} D_i, \mathbb{Q})$ such that:

(a) It contains all the functionals of the form $\bigvee_{i\leqslant n} \theta_i(g)$ for every type $g \in F$. In particular, we declare, for every $\gamma \in D_0$ and $i = 0, 1$

$$h_\gamma^{(p)} := \bigvee_{j\leqslant n} \theta_j(g) \tag{63}$$

where $g \in F$ is the type of $h_\gamma^{(p_0)}$.

(b) Let $1 \leqslant i \leqslant n$ and let $\gamma \in D_i \setminus R$. Then F_p contains $h_\gamma^{(p_i)}$ and we declare

$$h_\gamma^{(p)} := h_\gamma^{(p_i)}.$$

It is clear that $p \in \mathbb{P}_4$ and that $p \leqslant p_i$ for every $i \leqslant n$. The proof that for every $f_0, \dots, f_{m-1} \in B_{(X_p)^*}$ there is $1 \leqslant i \leqslant n$ such that $\max_{l<k, j<m} |f_j(v_0^{(l)}) - f_j(v_i^{(l)})| \leqslant \varepsilon$ is quite similar to the corresponding proof in Theorem 5.6. We leave the details to the reader. □

6. Schauder and Markushevich bases

Recall that a sequence $(x_\alpha)_{\alpha<\kappa}$ indexed by an ordinal κ in a Banach space X is called a K-basic sequence if for every sequence of scalars $(a_\alpha)_{\alpha<\kappa}$ and every ordinal $\gamma < \kappa$ one has that

$$\left\| \sum_{\alpha<\gamma} a_\alpha x_\alpha \right\|_X \leqslant K \left\| \sum_{\alpha<\kappa} a_\alpha x_\alpha \right\|_X.$$

The basic sequence is called monotone when $K = 1$. It is clear that if $(x_\alpha)_{\alpha<\kappa}$ is K-basic, then for every $\alpha < \kappa$ there is a functional x_α^* in X^* such that $x_\alpha^*(x_\beta) = \delta_{\alpha,\beta}$ for every $\beta < \kappa$. Hence, $(x_\alpha, x_\alpha^*)_{\alpha<\kappa}$ is a biorthogonal sequence.

The sequence $(x_\alpha)_{\alpha<\kappa}$ is called a K-(Schauder) basis of X if it is a K-basic sequence that in addition is fundamental in X, i.e. $\langle x_\alpha\rangle_{\alpha<\kappa}$ is dense in X. It is easy to see that if $(x_\alpha)_{\alpha<\kappa}$ is a K-basis of X, then $(x_\alpha, x_\alpha^*)_\alpha$ is fundamental and total, i.e. it is a Markushevich basis. The converse is not always true. Indeed, while every separable Banach space has always a Markushevich basis, there are separable Banach spaces without Schauder bases. The corresponding problem for non-separable spaces is still open.

Problem 3. Is it true that if a Banach space has an uncountable Markushevich basic sequence, then it has an uncountable basic sequence?

While the generic spaces that we could describe so far are not sufficient for solving this problem it is still easy for us to produce examples of generic spaces with ε-biorthogonal fundamental and total sequences yet having no uncountable basic sequences, and in fact, having no uncountable biorthogonal sequences. We start with a description of one such example.

Definition 6.1. Let $\varepsilon > 0$. Let $\widetilde{\mathbb{P}}_0$ be the forcing consisting on all conditions $p = (D_p, F_p, A_p, H_p)$ in $\mathbb{P}_0(\varepsilon)$ (see Definition 4.3) such that $A_p = D_p$.

Then it is easy to see that $\widetilde{\mathbb{P}}_0$ has the (AMP), but not the (EP). Still, it can be proved that, given a generic filter $\mathbb{G}$ for $\widetilde{\mathbb{P}}_0$, the sequence $(h_\gamma^{(\mathbb{G})})_{\gamma<\omega_1}$ is ε-equivalent to the unit basis of $\ell_1(\omega_1)$. It readily follows that $(u_\gamma, h_\gamma^{(\mathbb{G})})_{\gamma<\omega_1}$ is a total and fundamental ε-biorthogonal in $X_{\mathbb{G}}$, yet $X_{\mathbb{G}}$ does not have uncountable basic sequences (indeed no uncountable biorthogonal sequences).

On the other hand, our method is appropriate to distinguish existence of uncountable K-basic sequences for different constants K, or in other words, for answering the following two natural questions.

Question 5. Is it true that if X has an uncountable K-basic sequence then it has an uncountable K'-basic sequence for some $K' < K$?

Question 6. Is it true that if X has an uncountable $1+\varepsilon$-basic sequences for every $\varepsilon > 0$, then it has an uncountable monotone basic sequence?

6.1. Distinguishing basic constants of bases

We describe here a generic Banach space in order to answer Question 5.

Definition 6.2. Let $K \geqslant 1$ be a rational number. We define the following partial ordering $\mathbb{P}_5 = \mathbb{P}_5(K)$. The conditions are (D_p, F_p, A_p, H_p) such that

(1) $A_p = \emptyset$,
(2) $h_\gamma^{(p)} = u_\gamma$ for every $\gamma \in D_p$,
(3) $F_p \subseteq K \cdot \mathrm{conv}_{\mathbb{Q}}(F_p)$.

To make the notation easier given a condition $p = (D_p, F_p, A_p, H_p)$, we write $p = (D_p, F_p)$ to denote it, as A_p and H_p are a priori defined.

Observe that it readily follows from the definition that for every condition $p \in \mathbb{P}_5$ the sequence $(u_\gamma)_{\gamma \in D_p}$ is a K-basis of X_p dominating the c_0 unit basis, i.e. $\| \sum_{\gamma \in D_p} a_\gamma u_\gamma \|_p \geqslant \max_{\gamma \in D_p} |a_\gamma|$ for every sequence of scalars $(a_\gamma)_{\gamma \in D_p}$.

It is not true that $\mathbb{P}_5$ has (EP) or (AMP). On the other hand, we have the following.

Proposition 6.3. *The poset $\mathbb{P}_5$ has the Shanin property and $D_{\mathbb{G}} = \omega_1$ for every generic filter $\mathbb{G}$ of $\mathbb{P}_5$. Hence, $(u_\gamma)_{\gamma < \omega_1}$ is an uncountable K-basis of the corresponding generic space $X_{\mathbb{G}}$.*

Proof. Fix a Δ-system $(p_\alpha)_{\alpha < \omega_1}$ of type $t = (N, F)$ and root R, fix an integer k, and $\alpha_0 < \cdots < \alpha_{k-1}$. For each $i < k$, let $\theta_i : N \to D_i$ be the unique order-preserving bijection. We prove that $p_{\alpha_0}, \ldots, p_{\alpha_{k-1}}$ are compatible by defining a corresponding amalgamation $p = (\bigcup_{i<k} D_i, F_p)$ in $\mathbb{P}_5$ as follows. F_p is the minimal convex subset of $c_{00}(\bigcup_{i<k} D_i)$ that contains:

(a) $\bigvee_{i<k} \theta_i(h)$ for every type $h \in F$.
(b) $(1/K)(\bigvee_{i<k} \theta_i(h)) \restriction \gamma$ for every $\gamma < \omega_1$.

Then $(D_p, F_p, D_p, \{u_\gamma\}_{\gamma \in D_p})$ is basic such that $p \leqslant p_{\alpha_i}$ for every $i < k$. Moreover it is easy to see that $p \in \mathbb{P}_5$.

To see that $D_{\mathbb{G}}$ uses that given a condition $p \in \mathbb{P}_5$ such that $\gamma \notin D_p$, the condition $q = (D_p \cup \{\gamma\}, F_p \cup \{u_\gamma\})$ is in $\mathbb{P}_5$ and $q \leqslant p$.

Finally, it is clear that $(u_\gamma)_{\gamma < \omega_1}$ is a K-basis of the corresponding generic Banach space. □

Fix a generic filter $\mathbb{G}$ of $\mathbb{P}_5$.

Theorem 6.4. *The sequence $(u_\gamma)_{\gamma < \omega_1}$ is a normalized K-basis of $X_{\mathbb{G}}$, and the space $X_{\mathbb{G}}$ does not have uncountable K'-basic sequences for $1 \leqslant K' < K$.*

Proof. We prove that $X_{\mathbb{G}}$ does not have uncountable K'-basic sequences for $1 \leqslant K' < K$. We may assume that K' is a rational number. Working towards a contradiction, suppose that there is an uncountable normalized K'-basic sequence. Let $K'' = (K + K')/2$, and let $n \in \mathbb{N}$ be such that

$$\frac{n}{K} + 1 < \frac{n}{K''}. \tag{64}$$

A simple approximation argument gives that there is a normalized sequence $(x_\alpha)_{\alpha < \omega_1}$ such that for every increasing $(\alpha_i)_{i<2n}$ one has that

$$\left\| \sum_{i<n} x_{\alpha_i} \right\| \leqslant K'' \left\| \sum_{i<n} x_{\alpha_i} - \sum_{i=n}^{2n-1} x_{\alpha_i} \right\|. \tag{65}$$

Let $\mathfrak{P}(v_0, \ldots, v_{2n-1})$ be the following configuration: Either $\|v_i\| \neq 1$ for some $i < 2n$ or

$$\left\| \sum_{i<n} v_i \right\| > K'' \left\| \sum_{i<n} v_i - \sum_{i=n}^{2n-1} v_i \right\|. \tag{66}$$

Then $\mathfrak{P}$ is unavoidable for $\mathbb{P}_5$, and hence (65) is impossible. We use the theorem to prove that $\mathfrak{P}$ is unavoidable. Let $(p_i, v_i)_{i<2n}$ be a Δ-system of type $t = (N, F, v)$ and root R. Let $\theta_i : N \to D_i$ be the corresponding order-preserving bijection. We may assume that $\|v\|_t = 1$ since otherwise, any amalgamation of $(p_i)_{i<2n}$ will satisfy the configuration $\mathfrak{P}$. Let $p = (\bigcup_{i<2n} D_i, F_p)$ be the condition we exposed above to prove that $\mathbb{P}_5$ has the Shanin property. We check that $\mathfrak{P}$ holds in X_p: Since $\|v\|_t = 1$, there is some $h \in F$ such that $|h(v)| = 1$. Hence

$$n \geqslant \left\| \sum_{i<n} v_i \right\|_p \geqslant \left| \bigvee_{i<2n} \theta_i(h)\left(\sum_{i<n} v_i\right)\right| = \sum_{i<n} \left|h(v)\right| = n. \tag{67}$$

Now suppose that $f \in F_p$ is as in (a) (in the proof of Proposition 6.3), i.e. $f = \bigvee_{i<2n} \theta_i(g)$ for some type $g \in F$. Then

$$f\left(\sum_{i<n} v_i - \sum_{i=n}^{2n-1} v_i\right) = ng(v) - ng(v) = 0. \tag{68}$$

Suppose now that $f = (1/K) \bigvee_{i<2n} \theta_i(g) \restriction \gamma$ for some $\gamma < \omega_1$ and some $g \in F$. Without loss of generality we may assume that $\gamma \in D_{p_{\alpha_{i_0}}}$ for some $i_0 < 2n$. If $\gamma \in R$, then

$$f\left(\sum_{i<n} v_i - \sum_{i=n}^{2n-1} v_i\right) = 0. \tag{69}$$

So, suppose that $\gamma \notin R$, and let $\delta \in N$ be such that $\theta_{i_0}(\delta) = \gamma$. The first case we treat is when $i_0 < n$. Then

$$\left| f\left(\sum_{i<n} v_i - \sum_{i=n}^{2n-1} v_i\right)\right| = \left| \frac{1}{K} \sum_{i<i_0} \theta_i g(v_i) + \left(\frac{1}{K}(g \restriction \delta)\right)(v)\right| \tag{70}$$

$$\leqslant \frac{1}{K}(\#L - 1) + \|v\|_t \leqslant \frac{n-1}{K} + 1 < \frac{n}{K''}, \tag{71}$$

the last inequality because of (64). Suppose now that $i_0 \geqslant n$. Then one has

$$\begin{aligned} \left| f\left(\sum_{i<n} v_i - \sum_{i=n}^{2n-1} v_i\right)\right| &= \frac{1}{K}\bigg| \left(\sum_{i<n-1} \theta_i g(v_i) - \sum_{n\leqslant i<i_0} \theta_i g(v_i)\right) \\ &\quad + \theta_{n-1}(g)(v_{n-1}) - \theta_{i_0}(g) \restriction \gamma(v_{i_0})\bigg| \\ &= \frac{1}{K}\left|\big(n-1-(i_0-n)\big) + g(v) - (g \restriction \delta)(v)\right| \\ &\leqslant \frac{n-1}{K} + \frac{1}{K}\big(\|v\| + \|v \restriction \delta\|\big) \leqslant \frac{n-1}{K} + \frac{1}{K}(1+K) = \frac{n}{K} + 1 \\ &< \frac{n}{K''} = \frac{1}{K''}\left\| \sum_{i<n} v_i \right\|_p. \end{aligned}$$

Finally, if $f = u_\gamma$ for some $\gamma \in \bigcup_{i<2n} D_{p_i}$, then one has that

$$\left| f\left(\sum_{i<n} v_i - \sum_{i=n}^{2n-1} v_i \right) \right| \leqslant \begin{cases} 0 & \text{if } \gamma \in R, \\ \|v\| \leqslant 1 & \text{if } \gamma \notin R. \end{cases}$$

It follows from all this that

$$\left\| \sum_{i<n} v_i - \sum_{i=n}^{2n-1} v_i \right\|_p < \frac{1}{K''} \left\| \sum_{i<n} v_i \right\|_p, \tag{72}$$

which proves that the configuration $\mathfrak{P}(v_i)_{i<2n}$ holds in X_p. □

6.2. Long monotone basic sequences

We now give an example of generic space which has a Schauder basis, which has an uncountable $1+\varepsilon$-basic sequences for every $\varepsilon > 0$, but which has no uncountable monotone basic sequences.

Definition 6.5. Let $\mathbb{P}_6$ be the set of all conditions $p = (D_p, F_p, A_p, H_p)$ such that:

(1) $A_p = D_p$ and for every $\gamma \in D_p$ one has that $(h_\gamma^{(p)})_\gamma = (1/n) \cdot u_\gamma$ for some $n \geqslant 1$. Set

$$N_p := \{ n \geqslant 1 \colon A_p^{(n)} \neq \emptyset \}.$$

(2) $u_\gamma \in F_p$ for every $\gamma \in D_p$.
(3) $F_p \setminus \{u_\gamma\}_{\gamma \in D_p}$ can be partitioned into disjoint pieces $\{G_n^{(p)}\}_{n \in N_p}$ such that for every n one has that:
  (3.1) $\{h_\gamma^{(p)} \colon \gamma \in A_p^{(1/n)}\} \subseteq G_n^{(p)} \subseteq \langle u_\gamma \colon \gamma \in A_p^{(1/n)} \rangle$.
  (3.2) For every $f \in G_n^{(p)}$ one has that $\|f\|_\infty \leqslant 1/n$.
  (3.3) $G_n^{(p)}$ 1-norms $\langle u_\gamma \colon \gamma \in A_p^{(1/n)} \rangle$.
  (3.4) $G_n^{(p)} \subseteq (n+1)/n \cdot \mathrm{conv}_{\mathbb{Q}}(F_p)$.

Since $A_p = D_p$ always, we write $p = (D_p, F_p, H_p)$ to denote the condition (D_p, F_p, D_p, H_p) in $\mathbb{P}_6$.

Proposition 6.6. *The forcing notion $\mathbb{P}_6$ has the Shanin property.*

Proof. Let $(p_\alpha)_{\alpha<\omega_1}$ be a Δ-system of type $t = (N, F, H)$ and root R, fix an integer k, and $\alpha_0 < \cdots < \alpha_{k-1}$. Let $\{G_n\}_{n \in N_t}$ be the partition of $F \setminus \{u_\gamma\}_{\gamma \in N}$ as in (3) above. For each $i < k$, let $\theta_i : N \to D_i$ be the unique order-preserving bijection. We define the following amalgamation $p = (\bigcup_{i<k} D_i, F_p, H_p)$ of $(p_{\alpha_i})_{i<k}$ in $\mathbb{P}_6$. Let F_p be the minimal symmetric subset of $c_{00}(\bigcup_{i<k} D_i)$ having the following elements:

(a) $\bigvee_{i<k} \theta_i(h)$ for every type $h \in F$.
(b) $(n/(n+1))(\bigvee_{i<k} \theta_i(h)) \restriction \gamma$ for every $h \in G_n$ and $\gamma < \omega_1$.

(c) $h_\gamma^{(p_i)}$ for every $i < k$ and $\gamma \in D_i$.
(d) u_γ for every $\gamma \in \bigcup_{i<k} D_i$.

For each $\gamma \in A_i$, $i < k$, we declare

$$h_\gamma^{(p)} := h_\gamma^{(p_i)}.$$

Let us call this condition p the *basic amalgamation* of $(p_{\alpha_i})_{i<k}$. It is clear that p is a basic condition such that $p \leqslant p_{\alpha_i}$ for every $i < k$. We rapidly check that $p \in \mathbb{P}_6$: Properties (1) and (2) are easy to verify. For each $n \in N_t$ we define

$$G_n^{(p)} := \left\{\bigvee_{i<k} \theta_i(g)\colon\ g \in G_n\right\} \cup \left\{\frac{n}{n+1}\bigvee_{i<k} \theta_i(g) \upharpoonright \gamma\colon\ g \in G_n,\ \gamma < \omega_1\right\} \cup \left\{h_\gamma^{(p)}\colon\ \gamma \in A_p^{(\frac{1}{n})}\right\}.$$

We leave to the reader the details of the proof that $\{G_n^{(p)}\}_{n\in N_t}$ fulfills (3). $\square$

From now on we fix a generic filter $\mathbb{G}$ of $\mathbb{P}_6$. Recall that $A_{\mathbb{G}}^{(1/n)} = \bigcup_{p\in\mathbb{G}} A_p^{(1/n)}$ for $n \in \mathbb{N}$.

Proposition 6.7. *For every $n \in \mathbb{N}$ the sequence $(u_\gamma\colon\ \gamma \in A_{\mathbb{G}}^{(1/n)})$ forms an uncountable $(n+1)/n$-basic normalized sequence in $X_{\mathbb{G}}$.*

Proof. Because of the property (3.2) and (3.3) one has that for every $n \in N$ the sequence $(u_\gamma)_{\gamma\in A_{\mathbb{G}}^{(1/n)}}$ is an $(n+1)/n$-basic sequence, while (2) gives that $\|u_\gamma\| = 1$ for every γ. We prove that for every $n \in \mathbb{N}$ the set $A_{\mathbb{G}}^{(1/n)}$ is uncountable. To see this, given $p \in \mathbb{P}_6$, $n \in \mathbb{N}$ and $\gamma < \omega_1$ we define

$$\mathcal{D}_{p,n,\gamma} := \left\{q \in \mathbb{P}_6\colon \text{ either } q \perp p \text{ or } q \leqslant p \text{ and } \max A_q^{(1/n)} \geqslant \gamma\right\}.$$

Then the desired results follow from the fact that $\mathcal{D}_{p,n,\gamma}$ is dense, so let us prove that. Let $r \leqslant p$ be a condition in $\mathbb{P}_6$. Let $\gamma_0 > \gamma, \max D_r$. Define

$$q = \left(D_r \cup \{\gamma_0\}, F_r \cup \left\{\pm u_{\gamma_0}, \pm\frac{1}{n}u_{\gamma_0}\right\}, H_r \cup \left\{h_{\gamma_0}^{(q)} := \frac{1}{n}u_{\gamma_0}\right\}\right).$$

Then q extends r and is in $\mathcal{D}_{p,n,\gamma}$. $\square$

Theorem 6.8. *The generic space $X_{\mathbb{G}}$ does not have uncountable monotone basic sequences.*

Proof. Going towards a contradiction we assume that $X_{\mathbb{G}}$ has an uncountable monotone basic sequence $(x_\alpha)_{\alpha<\omega_1}$. Without loss of generality, we assume that it is normalized. We now use Proposition 3.7(d) for $(x_\alpha)_\alpha$ and $\varepsilon = 1/4$ to find $0 < \delta < 1$ and $\Gamma \subseteq \omega_1$ uncountable such that for every $f \in B_{(X_{\mathbb{G}})^*}$ and every $\alpha \in \Gamma$ if $\|f\|_\infty \leqslant \delta$ then $|f(x_\alpha)| \leqslant 1/3$. Let $\bar{n} \in \mathbb{N}$ be the integer part of $1/\delta$. Let also n be such that

$$\frac{\bar{n}}{\bar{n}+1}n + 1 < \frac{2\bar{n}}{2\bar{n}+1}n. \tag{73}$$

Finally, we find a normalized sequence $(y_\alpha)_{\alpha<\omega_1}$ such that:

1. For every finite subsets $s<t$ of ω_1 such that $\#s=\#t=n$ one has that

$$\Big\|\sum_{\alpha\in s} y_\alpha\Big\|_{\mathbb{G}} \leqslant \Big(1+\frac{1}{2\bar{n}}\Big)\Big\|\sum_{\alpha\in s} y_\alpha-\sum_{\alpha\in t} y_\alpha\Big\|_{\mathbb{G}}. \tag{74}$$

2. For every $\alpha<\omega_1$ and every $f\in B_{(X_{\mathbb{G}})^*}$ if $\|f\|_\infty\leqslant 1/\bar{n}$ then $|f(x_\alpha)|\leqslant 1/3$.

Let $\mathfrak{P}(v_0,\dots,v_{2n-1})$ be the following configuration on the points $(v_i)_{i<2n}$ in some space $X\subseteq c_{00}(\omega_1)$:

(I) Either $\|v_i\|_X\neq 1$ for some $i<2n$ or there is $f\in B_{X^*}$ with $\|f\|_\infty\leqslant 1/\bar{n}$ and $|f(v_i)|>1/3$, or else

(II)

$$\Big\|\sum_{i<n} v_i\Big\|_X > \Big(1+\frac{1}{2\bar{n}}\Big)\Big\|\sum_{i<n} v_i-\sum_{i=n}^{2n-1} v_i\Big\|_X. \tag{75}$$

We prove that $\mathfrak{P}$ is unavoidable for $\mathbb{P}_6$, and so the existence of the sequence $(y_\alpha)_\alpha$ with the properties 1. and 2. above is impossible. Let $(p_i,v_i)_{i<2n}$ be a Δ-system of type $t=(N,F,H,v)$ and root R. For each $i<2n$ let $\theta_i:N\to D_i$ be the unique order-preserving. Let p be the basic amalgamation of $(p_i)_{i<2n}$ exposed in the proof of Proposition 6.6. If $\|v\|_t\neq 1$ or if there is some $f\in B_{X_t^*}$ with $\|f\|_\infty\leqslant 1/\bar{n}$ and $|f(v)|>1/3$, then (I) holds in X_p. Suppose then that $\|v\|_t=1$ and $|f(v)|\leqslant 1/3$ for every $f\in B_{X_t^*}$ with $\|f\|_\infty\leqslant 1/\bar{n}$. We check that (II) holds in X_p: First of all, we see that

$$\Big\|\sum_{i<n} v_i\Big\|_p = n. \tag{76}$$

It is clear that $\|\sum_{i<n} v_i\|_p\leqslant n$. Now let $g\in F$ be such that $|g(v)|=1$. Then $f=\bigvee_{i<2n}\theta_i(g)\in F_p$ and so,

$$n=\Big|f\Big(\sum_{i<n} v_i\Big)\Big|\leqslant\Big\|\sum_{i<n} v_i\Big\|_p\leqslant n. \tag{77}$$

We check now that

$$\Big\|\sum_{i<n} v_i-\sum_{i=n}^{2n-1} v_i\Big\|_p<\frac{2\bar{n}}{2\bar{n}+1}n: \tag{78}$$

Set $w=\sum_{i<n} v_i-\sum_{i=n}^{2n-1} v_i$. Let $g\in F$, and let $f=\bigvee_{i<2n}\theta_i(g)$. Then

$$f(w)=ng(v)-ng(v)=0. \tag{79}$$

Suppose that $f=\delta u_\gamma$ for some $\gamma\in D_{p_i}$, $i<2n$ and $|\delta|\leqslant 1$. Then

$$\left|f(w)\right|=\begin{cases}|(v_i)_\gamma|\leqslant 1 & \text{if } \gamma\notin R,\\ 0 & \text{if } \gamma\in R.\end{cases} \tag{80}$$

Finally suppose that $f=(k/(k+1))\bigvee_{i<2n}\theta_i(g)\restriction\gamma$ for some $k\in M_t$, $g\in G_k^{(t)}$ and $\gamma\in D_p$. We distinguish two cases:

CASE 1. $k\geqslant\bar n$. Then, by the property (3.1) of $G_k^{(t)}$, it follows that

$$\left\|\frac{k}{k+1}g\restriction\gamma\right\|_\infty\leqslant\|g\|_\infty\leqslant\frac{1}{n}\leqslant\frac{1}{\bar n}. \tag{81}$$

Let $i_0<2n$ be such that $\gamma\in D_{p_{i_0}}$, and let $\delta\in N$ be such that $\theta_{i_0}(\delta)=\gamma$. If $\gamma\in R$, then it follows that $f(w)=0$. Assume now $\gamma\notin R$. Then from the negation of (I) for t, we get

$$\left|g(v)\right|,\left|\left(\frac{k}{k+1}g\restriction\delta\right)(v)\right|\leqslant\frac{1}{3}. \tag{82}$$

Hence,

$$\left|f(w)\right|\leqslant\sum_{i<i_0}\frac{k}{k+1}\left|\theta_i(g)(v_i)\right|+\left|\left(\frac{k}{k+1}\theta_{i_0}(g)\restriction\gamma\right)(v_{i_0})\right|\leqslant 2n\frac{1}{3}<\frac{2\bar n}{2\bar n+1}n. \tag{83}$$

CASE 2. $k<\bar n$. Then, as in the corresponding part of the proof of Theorem 6.4 one has that

$$\begin{aligned}\left|f(w)\right|&=\frac{k}{k+1}\left|\left(\sum_{i<n-1}\theta_i g(v_i)-\sum_{n\leqslant i<i_0}\theta_i g(v_i)\right)+\theta_{n-1}(g)(v_{n-1})-\theta_{i_0}(g)\restriction\gamma(v_{i_0})\right|\\ &=\frac{k}{k+1}\left|\big(n-1-(i_0-n)\big)+g(v)-(g\restriction\delta)(v)\right|\\ &\leqslant\frac{k(n-1)}{k+1}+\frac{k}{k+1}\big(\|v\|+\|v\restriction\delta\|\big)\leqslant\frac{k(n-1)}{k+1}+\frac{k}{k+1}\frac{2k+1}{k}=\frac{k}{k+1}n+1\\ &<\frac{\bar n}{\bar n+1}n+1<\frac{2\bar n}{2\bar n+1}n.\qquad\square\end{aligned}$$

It could be seen that the generic spaces of this sections are not Gurarij, not even Lindenstrauss spaces. So, we are lead to the following natural question.

Problem 4. Does there exist a Lindenstrauss space with a K-basis but with no K'-basic sequences for any $K'<K$?

7. Some general properties of generic spaces

For the next notions we use the following notation: Given a forcing notion $\mathbb{P}$, $\delta\in\Delta_{\mathbb{P}}$ and $a>0$ we define

$$(\delta)_\infty := \sup\bigl\{\bigl\|h_\gamma^{(p)}\bigr\|_\infty\colon\ p \in \mathbb{P},\ \gamma \in A_p^{(\delta)}\bigr\},$$
$$\Delta_{\mathbb{P}}^{(\geqslant a)} := \bigl\{\delta \in \Delta_{\mathbb{P}}\colon (\delta)_\infty \geqslant a\bigr\}.$$

Note that $(\delta)_\infty \geqslant |\delta|$, hence, $\delta \in \Delta_{\mathbb{P}}^{(\geqslant|\delta|)}$ for every $\delta \in \Delta_{\mathbb{P}}$.

Definition 7.1. We say that a forcing notion $\mathbb{P}$ has the *Strong Amalgamation Property* (SAMP) if for every $a > 0$ there is $\varepsilon_a \in]0,1] \cap \mathbb{Q}$ such that whenever $(p_i)_{i<k}$ is a Δ-system in $\mathbb{P}$ of type $t = (N, F, A, H)$ and root R and $p = (D_p, F_p, A_p, H_p)$ is a pre-amalgamation of $(p_i)_{i<k}$ such that:

(a) $F_p \setminus \pm H_p = \{\bigvee_{i<k} \theta_i(g)\colon g \in F \setminus \pm H\}$,
(b) $h_\gamma^{(p)} = \bigvee_{i<k} h_\gamma^{(p_i)}$ for every $\gamma \in R$,
(c) $h_\gamma^{(p)} = h_\gamma^{(p_i)}$ for every $\gamma \in D_{p_i} \setminus R$ with $(h_\gamma^{(p_i)})_\gamma \notin \Delta_{\mathbb{P}}^{(\geqslant a)}$,
(d) $h_\gamma^{(p)} = h_\gamma^{(p_i)} \vee \bigvee_{i<j<k} \varepsilon_j \cdot \theta_j(h_j)$ for every $\gamma \in A_i \setminus R$ with $(h_\gamma^{(p_i)})_\gamma \in \Delta_{\mathbb{P}}^{(\geqslant a)}$, and where $h_j \in \mathrm{conv}_{\mathbb{Q}}(\pm H)$ is such that $h_j \upharpoonright |R| = 0$ for every $i < j < k$, and

$$|\varepsilon_j| \leqslant \varepsilon_a, \tag{84}$$

then p is in $\mathbb{P}$.

Remark 7.2.

(a) If $\mathbb{P}$ has (SAMP) then so does $\mathbb{P}_H$.
(b) The forcing notions $\mathbb{P}_{\mathrm{basic}}, \mathbb{P}_0, \dots, \mathbb{P}_3$ have (SAMP), and $\mathbb{P}_4, \mathbb{P}_5, \mathbb{P}_6$ don't. We see that more than a formal there is a geometrical reason for that.

7.1. Corson property and support sets

Recall that a Banach space X has the Corson property (C) if every family of closed convex subsets of X that has empty intersection contains a countable subfamily with the same property. Thus the property (C) is a natural convex analogue of the Lindelöf property of the weak topology of X. It is clearly weaker than this property since every closed convex sets is closed relative to the weak topology. Recall also that a *support set* in X is a nonempty closed convex subset C of X which is supported by all of its points, or in other words if for every $x \in C$ there is $f \in X^*$ such that $f(x) = \min_{y\in C} f(y) < \sup_{y\in C} f(y)$. It is not difficult to see [3] that the existence of such sets in X is equivalent to the existence of an uncountable *semi-biorthogonal system*, a sequence $(x_\alpha, f_\alpha)_{\alpha<\omega_1}$ of elements of $X \times X^*$ such that

$$f_\beta(x_\beta) = 1, \qquad f_\beta(x_\alpha) = 0 \quad \text{for } \alpha < \beta \quad \text{and} \quad f_\beta(x_\alpha) \geqslant 0 \quad \text{for } \alpha > \beta.$$

Theorem 7.3. *Suppose that $\mathbb{P}$ is a forcing condition with the* (SAMP). *Then any of its generic spaces has the Corson property* (C) *and fails to have support sets.*

It follows that $\mathbb{P}_4, \mathbb{P}_5$ and $\mathbb{P}_6$ do not have (SAMP) because the corresponding generic spaces do have support sets.

Proof of Theorem 7.3. Fix the sequence $(\varepsilon_a)_{a>0}$ witnessing that $\mathbb{P}$ has (SAMP). We prove first that $X_{\mathbb{G}}$ does not have uncountable semi-biorthogonal sequences. Suppose otherwise that $(y_\alpha)_{\alpha<\omega_1}$ is a semi-biorthogonal normalized sequence in $X_{\mathbb{G}}$. Then there is some $\bar{k} \in \mathbb{N}$, $\bar{k} \geqslant 2$ and some uncountable sequence $(f_\alpha)_{\alpha\in\Gamma}$ of functionals in $B_{X_{\mathbb{G}}^*}$ such that

$$f_\alpha(y_\alpha) \geqslant \frac{3}{\bar{k}}, \qquad f_\alpha(y_\beta) = 0 \quad \text{and} \quad f_\beta(y_\alpha) \geqslant 0 \quad \text{for every } \beta < \alpha \text{ in } \Gamma.$$

This implies that if $A < \bar{\alpha} < B$ are finite subsets of Γ it follows that

$$\Big\| -\sum_{\alpha\in A} y_\alpha + \bar{k} y_{\bar{\alpha}} + \sum_{\alpha\in B} y_\alpha \Big\|_{X_{\mathbb{G}}} \geqslant f_{\bar{\alpha}}\Big(-\sum_{\alpha\in A} y_\alpha + \bar{k} y_\alpha + \sum_{\alpha\in B} y_\alpha \Big) \geqslant \frac{3\bar{k}}{\bar{k}} \geqslant 3.$$

We use now Proposition 3.7(d) to the sequence $(y_\alpha)_{\alpha\in\Gamma}$ and $\varepsilon = 1/(2\bar{k})$ to find a rational number $\bar{\delta} > 0$ and an uncountable $\bar{\Gamma} \subseteq \Gamma$ such that for every $f \in B_{(X_{\mathbb{G}})^*}$ with $\|f\|_\infty \leqslant \bar{\delta}$ then $|f(y_\alpha)| \leqslant 1/(2\bar{k})$ for every $\alpha \in \bar{\Gamma}$. Let $\bar{m}$ be an integer such that

$$\bar{m} \cdot \varepsilon_{\bar{\delta}} \geqslant 1. \tag{85}$$

Now let $(x_\alpha)_{\alpha\in\bar{\Gamma}}$ be a normalized sequence of elements of $c_{00}(\omega_1, \mathbb{Q})$ such that

(a) for every $A < \bar{\alpha} < B$ in $\bar{\Gamma}$ with $\#(A) = \bar{k} \cdot (\bar{m}+1)$ and $\#(B) = \bar{k} \cdot \bar{m}$ one has that

$$\Big\| -\sum_{\alpha\in A} x_\alpha + \bar{k} \cdot x_{\bar{\alpha}} + \sum_{\alpha\in B} x_\alpha \Big\|_{X_{\mathbb{G}}} > 2. \tag{86}$$

(b) For every $f \in B_{(X_{\mathbb{G}})^*}$ with $\|f\|_\infty \leqslant \bar{\delta}$ and every $\alpha \in \bar{\Gamma}$ one has that $|f(x_\alpha)| \leqslant 1/\bar{k}$.

We are going to see that such sequence (x_α) does not exist. To prove this, let $\mathfrak{P}(v_0, \dots, v_{\bar{k}(2\bar{m}+1)})$ be the following configuration on the points $(v_i)_{i\leqslant\bar{k}(2\bar{m}+1)}$ in a space $X \subseteq c_{00}(\omega_1)$:

(I) Either there is $f \in B_{X^*}$ with $\|f\|_\infty \leqslant \bar{\delta}$ and $i < \bar{k}(2\bar{m}+1)$ such that $|f(v_i)| > 1/\bar{k}$, or else
(II)

$$\Big\| -\sum_{i=0}^{\bar{k}(\bar{m}+1)-1} v_i + \bar{k} \cdot v_{\bar{k}(\bar{m}+1)} + \sum_{i=\bar{k}(\bar{m}+1)+1}^{\bar{k}(2\bar{m}+1)} v_i \Big\|_X \leqslant 2 \max_{i\leqslant\bar{k}(2\bar{m}+1)} \|v_i\|. \tag{87}$$

It follows by (a) and (b) that the $\mathfrak{P}$ is *not* unavoidable for the sequence $(x_\alpha)_{\alpha\in\Delta}$. This is impossible by the following.

Claim 7.3.1. *The configuration $\mathfrak{P}$ is unavoidable for $\mathbb{P}$.*

Proof. We use Theorem 3.36. Set $L = \bar{k}(2\bar{m}+1)$, and let $(p_i, v_i)_{i\leqslant L}$ be a Δ-system of type $t = (N, F, A, H, v)$ and root R, $\theta_i : N \to D_i$ be the corresponding order-preserving bijections for $i \leqslant L$. We may assume that for every $g \in B_{(X_t)^*}$ with $\|g\|_\infty \leqslant \delta$ one has $|g(v)| \leqslant 1/\bar{k}$, since otherwise (I) holds in *any* amalgamation $p \in \mathbb{P}_1$ of $(p_i)_{i \leqslant L}$. After normalization, we may assume that $\|v\|_t = 1$. Let $p = (\bigcup_{i<L} D_i, F_p, \bigcup_{i<L} A_i, H_p)$ be the following basic conditions:

(c) F_p is the minimal symmetric subset of $c_{00}(\bigcup_{i<L} D_i, \mathbb{Q})$ such that:
  (c.0) It contains all the functionals of the form $\bigvee_{i \leqslant L} \theta_i(g)$ for every type $g \in F$.
  (c.1) Let $i \leqslant L$, $\gamma \in A_i \setminus R$, and let $h \in F$ be the type of $h_\gamma^{(p_i)}$.
    (c.1.0) If $i \neq \bar{k}(\bar{m}+1)$ or if $\gamma \in A_i^{(\delta)}$ with $(\delta)_\infty < \bar{\delta}$ then F_p contains

$$h_\gamma^{(p_i)}.$$

    (c.1.1) If $i = \bar{k}(\bar{m}+1)$, and either $\gamma \in D_i \setminus A_i$ or $\gamma \in A_i^{(\delta)}$ with $(\delta)_\infty \geqslant \bar{\delta}$, then F_p contains

$$h_\gamma^{(p_i)} \vee \bigvee_{j=\bar{k}(\bar{m}+1)+1}^{\bar{k}(2\bar{m}+1)} -\frac{1}{\bar{m}}\theta_j(h).$$

(d) For each $i \leqslant L$ and each $\gamma \in D_i$, if $h \in F$ denotes the type of $h_\gamma^{(p_i)}$, then we have that:
  (d.0) If $\gamma \in R$ then

$$h_\gamma^{(p)} := \bigvee_{i \leqslant L} \theta_i(h). \tag{88}$$

  (d.1) Suppose that $R < \gamma$, and that $i \neq \bar{k}(\bar{m}+1)$ or that $\gamma \in A_i^{(\delta)}$ with $(\delta)_\infty < \bar{\delta}$. Then

$$h_\gamma^{(p)} := h_\gamma^{(p_i)}.$$

  (d.2) Suppose that $R < \gamma$, $i = \bar{k}(\bar{m}+1)$ and, either $\gamma \in D_i \setminus A_i$ or $\gamma \in A_i^{(\delta)}$ with $(\delta)_\infty \geqslant \bar{\delta}$. Then

$$h_\gamma^{(p)} = h_\gamma^{(p_i)} \vee \bigvee_{j=\bar{k}(\bar{m}+1)+1}^{\bar{k}(2\bar{m}+1)} -\frac{1}{\bar{m}}\theta_j(h). \tag{89}$$

It follows that $p \in \mathbb{P}$, and that $p \leqslant p_i$ for every $i \leqslant L$.

Let us check (87): Set

$$w = -\sum_{i=0}^{\bar{k}(\bar{m}+1)-1} v_i + \bar{k} \cdot v_{\bar{k}(\bar{m}+1)} + \sum_{i=\bar{k}(\bar{m}+1)+1}^{\bar{k}(2\bar{m}+1)} v_i.$$

Suppose first that $f = \bigvee_{i \leqslant L} \theta_i(g)$ for some type $g \in F$. Then it follows that

$$f(w) = -\big(\bar{k}(\bar{m}+1)\big)g(v) + \bar{k}g(v) + \bar{k}\bar{m}g(v) = 0.$$

Suppose that $f = h_\gamma^{(p_i)}$ is as in (c.1.0). The first case to consider now is when $\gamma \in A_i^{(\delta)}$ and $(\delta)_\infty < \bar{\delta}$. By the definition of $\bar{\delta}$ and the negation of (I), one has that

$$\left|f(w)\right| = \left|(w)_i \cdot h_\gamma^{(p_i)}(v_i)\right| \leqslant 1,$$

because $(w)_i \leqslant \bar{k}$, and $|h_{(n,u_\gamma)}^{(p_i)}(v_i)| \leqslant 1/\bar{k}$, and where $(w)_i$ denotes the ith-coordinate of w with respect to $(v_j)_j$. The second case is when $i \neq \bar{k}(\bar{m}+1)$. It follows that

$$\left|f(w)\right| = \left|(w)_i \cdot h_\gamma^{(p_i)}(v_i)\right| \leqslant 1,$$

because in this case $(w)_i \leqslant 1$, and $|h_\gamma^{(p_i)}(v_i)| \leqslant \|v\|_t = 1$.

Finally, if f is as in (c.1.1), then $f = h_\gamma^{(p_{\bar{k}(\bar{m}+1)})} \vee \bigvee_{j=\bar{k}(\bar{m}+1)+1}^{\bar{k}(2\bar{m}+1)} (-1/\bar{m})\theta_j(h)$, where $R < x$ and $h \in F$ is the type of $h_\gamma^{(p_{\bar{k}(\bar{m}+1)})}$. Then

$$f(w) = \bar{k}h(v) - \frac{1}{\bar{m}}(\bar{k}\bar{m})h(v) = 0. \qquad \square$$

The proof of the Corson property (C) of the generic space $X_{\mathbb{G}}$ is quite similar, so we only sketch it. If $X_{\mathbb{G}}$ would not have the property (C), then one could find a sequence $(x_\alpha, f_\alpha)_{\alpha<\omega_1}$ of pairs of points and bounded functionals of $X_{\mathbb{G}}$ such that $f_\alpha(x_\alpha) = 1$, $f_\beta(x_\alpha) = 0$ and $f_\alpha(x_\beta) \leqslant 0$ for every $\alpha < \beta$. It follows that there exists an uncountable set $\Gamma \subseteq \omega_1$ and some integer $\bar{k}$ such that for every finite $A, B \subseteq \Gamma$ and $\bar{\alpha} \in \Gamma$ with $A < \bar{\alpha} < B$ one has that

$$\left\| \sum_{\alpha \in A} y_\alpha + \bar{k} y_{\bar{\alpha}} - \sum_{\alpha \in B} y_\alpha \right\|_{X_{\mathbb{G}}} \geqslant 3.$$

Arguing similarly as for the proof of non-existence of support sets, one can then find some uncountable normalized separated sequence $(x_\alpha)_{\alpha<\omega_1}$, and some integer $\bar{n}$ such that for every $A < \bar{\alpha} < B$ with $\#(A) = \bar{k} \cdot (k_{\bar{n}-1})$ and $\#(B) = \bar{k} \cdot (k_{\bar{n}-1} + 1)$ one has that

$$\left\| \sum_{\alpha \in A} x_\alpha + \bar{k} \cdot x_{\bar{\alpha}} - \sum_{\alpha \in B} x_\alpha \right\|_{X_{\mathbb{G}}} > 2. \tag{90}$$

Now, similarly as before, one can easily define an unavoidable configuration $\mathfrak{P}$ disproving the inequality in (90). We leave the details to the reader. $\square$

7.2. Block representability

Definition 7.4. Let κ, λ be two ordinals and let $(x_i)_{i<\kappa}$ and $(y_i)_{i<\lambda}$ be two sequences in some Banach space X. We say the $(y_i)_{i<\lambda}$ is a *block subsequence* of $(x_i)_{i<\kappa}$ if there is a block subsequence $(s_i)_{i<\lambda}$ of finite subsets of κ and scalars $(a_j)_{j\in s_i}$ $(i < \lambda)$ such that for every $i < \lambda$ one has that

$$y_i = \sum_{j\in s_i} a_j x_j.$$

Fix now $(v_i)_{i<n}$ a finite sequence in some other Banach space Y. We say that $(v_i)_{i<n}$ is *K-block representable* in $(x_i)_{i<\kappa}$ if there is block subsequence $(y_i)_{i<n}$ of $(x_i)_{i<\kappa}$ which is K-equivalent to $(v_i)_{i<n}$. We say that $(v_i)_{i<n}$ is *K^+-block representable* in $(x_i)_{i<\kappa}$ if for every $\varepsilon>0$ there is block subsequence $(y_i)_{i<n}$ of $(x_i)_{i<\kappa}$ which is $K+\varepsilon$-equivalent to $(v_i)_{i<n}$.

Theorem 7.5. *Let $\mathbb{P}$ be a forcing notion with the* (EAMP)*, and let $\mathbb{G}$ be a generic forcing for it. Then for every finite sequence is 1^+-block representable in any uncountable separated normalized sequence of points of $X_{\mathbb{G}}$, and in any uncountable separated normalized sequence of points of $X_{\mathbb{G},H}$.*

More precisely, if $(x_\alpha)_{\alpha<\omega_1}$ is a separated sequence in $X_{\mathbb{G}}$, and if $(v_i)_{i<k}$ is a finite basis, and $\varepsilon>0$ then there is a block sequence $(y_i)_{i<k}$ of $(x_\alpha)_{\alpha<\omega_1}$ such that

$$\big((y_i)_{i<k},\|\cdot\|_{\mathbb{G}}\big) \text{ is 1-equivalent to } \big(\big(\pi_{\mathbb{G}}(y_i)\big)_{i<k},\|\cdot\|_{\mathbb{G},H}\big) \text{ is } 1+\varepsilon\text{-equivalent to } (v_i)_{i<k}. \tag{91}$$

Remark 7.6. It follows from the previous result that under the assumption of (EAMP) that every finite sequence is 1^+-block representable in any uncountable separated normalized sequence of points of $X_{\mathbb{G},H}$, because $\pi_{\mathbb{G}}: X_{\mathbb{G}} \to X_{\mathbb{G},H}$ and pre-image of a separated sequence is also separated.

We start with the following result. Suppose that $\mathbb{P}$ has (EAMP).

Lemma 7.7. *Let $((e_i)_{i<n},\|\cdot\|_G)$ be a normalized $\mathbb{Q}$-basis of a $\mathbb{Q}$-f.d. space G, and let $(x_\alpha)_{\alpha<\omega_1}$ be any uncountable ε-separated normalized sequence in $X_{\mathbb{G}}$ consisting of points of $c_{00}(\omega_1,\mathbb{Q})$. Then for every integer k with*

$$k \geqslant \max\left\{\frac{1}{\varepsilon}, \max_{i<n}\|e_i^*\|_{G^*}\right\} \tag{92}$$

there exists a block sequence $(s_i)_{i<n}$ of finite subsets of ω_1 each one of size $2k^2$ such that the sequences

$$\big(y_i,\|\cdot\|_G\big)_{i<n} \quad \text{and} \quad \big(y_i,\|\cdot\|_{G,H}\big)_{i<n} \quad \text{are 1-equiv. to } (e_i)_{i<n}, \tag{93}$$

where for each $i<n$,

$$s_i = \big\{\alpha_0^{(i)}<\cdots<\alpha_{2k^2-1}^{(i)}\big\} \text{ is the increasing enumeration of } s_i \quad \text{and} \tag{94}$$

$$y_i := \frac{1}{k}\sum_{j<2k^2}(-1)^j x_{\alpha_j^{(i)}}. \tag{95}$$

Before we prove this lemma, we use it.

Proof of Theorem 7.5. Fix a sequence $(e_i)_{i<n}$ and a separated normalized sequence $(z_\alpha)_{\alpha<\omega_1}$ in $X_{\mathbb{G}}$. We fix a large enough k satisfying (92). Note now that if $x\in c_{00}(\omega_1)$ then

$$\|\pi_{\mathbb{G}} z_\alpha - x\|_{\mathbb{G},H} = \|\pi_{\mathbb{G}} z_\alpha - \pi_{\mathbb{G}} x\| \leqslant \|z_\alpha - x\|. \tag{96}$$

Let $\delta > 0$, and let $(x_\alpha)_\alpha$ be a normalized separated sequence in $c_{00}(\omega_1)$ such that $\|z_\alpha - x_\alpha\|_{\mathbb{G}} \leqslant \varepsilon$. Let T be the linear isomorphism $T : \langle x_\alpha\rangle_\alpha \to \langle z_\alpha\rangle_\alpha$ linearly extending $T(x_\alpha) = y_\alpha$. Then if $(y_i)_{i<n}$ is any block sequence of $(x_\alpha)_\alpha$ as in (95), it follows that for every sequence of scalars $(a_i)_{i<n}$ one has that

$$\left|\left\|\sum_{i<n} a_i y_i\right\|_{\mathbb{G},H} - \left\|\sum_{i<n} a_i \pi_{\mathbb{G}}\big(T(y_i)\big)\right\|_{\mathbb{G},H}\right|, \left|\left\|\sum_{i<n} a_i y_i\right\|_{\mathbb{G}} - \left\|\sum_{i<n} a_i T(y_i)\right\|_{\mathbb{G}}\right|$$
$$\leqslant \max_{i<n} |a_i| \cdot 2k^2 n. \tag{97}$$

It should be clear that for $\delta > 0$ small enough (97) together with (93) gives that $(T(y_i))_{i<n}$ and $(\pi_{\mathbb{G}}(T(y_i)))_{i<n}$ are $1+\varepsilon$-equivalent to $(e_i)_{i<n}$. $\square$

Proof of Lemma 7.7. Let $((e_i)_{i<n}, \|\cdot\|_G)$ be a $\mathbb{Q}$-basic sequence, $\varepsilon > 0$ and let $k \in \mathbb{N}$ be such that (92) holds. Define then the following metric configuration $\mathfrak{P}(v_i)_{i<2k^2n}$: Either

(a) $\|v_i\| = 1$ for all $i < 2k^2 n$ and $\min_{i<j<2k^2n} \|v_i - v_j\| \leqslant \varepsilon$, or else
(b) the sequences

$$\left(\left(\frac{1}{k}\left(\sum_{j=2k^2 i}^{2k^2(i+1)-1} (-1)^j v_j\right)\right)_{i<n}, \|\cdot\|_p\right) \quad \text{and} \quad \left(\left(\frac{1}{k}\left(\sum_{j=2k^2 i}^{2k^2(i+1)-1} (-1)^j v_j\right)\right)_{i<n}, \|\cdot\|_{p,H}\right)$$

are 1-equivalent to $(e_i)_{i<n}$.

The desired result readily follows from the fact that the configuration $\mathfrak{P}(v_i)_{i<2k^2n}$ is unavoidable for $\mathbb{P}$. In order to prove this, we use Theorem 3.36. So we fix a Δ-system $(p_i, v_i)_{i<2k^2n}$ of type $t = (N, F, A, H, v)$ and with root R. Let $\theta_i : N \to D_i$ be the corresponding order-preserving bijections.

CASE 1. Suppose first that

$$\text{either } \|v\|_t \neq 1 \quad \text{or } d_{t,H}\big(v, \langle u_i\rangle_{i\in|R|}\big) \leqslant \varepsilon. \tag{98}$$

Let p denote the basic amalgamation of $(p_i)_i$.

SUBCASE 1.1 $\|v\|_t \neq 1$. It follows that $\|v_i\|_p \neq 1$ for every $i < 2k^2 n$, so $\mathfrak{P}(v_i)_i$ holds in p.

SUBCASE 1.2 Suppose that

$$\|v\|_t = 1 \quad \text{and} \quad d_{t,H}\big(v, \langle u_i\rangle_{i\in|R|}\big) \leqslant \varepsilon. \tag{99}$$

Then, let us check that $\|v_i - v_j\|_p \leqslant \varepsilon$ for every $i, j < 2k^2 n$, which will also prove that $\mathcal{P}(v_i)_i$ holds in p: If $f = \bigvee_{i<2k^2n} \theta_i(h)$ for some $h \in F$, then it follows that $f(v_i - v_j) = h(v) - h(v) = 0$. Otherwise, $f = h_\gamma^{(p_i)}$ for some $i < 2k^2 n$ and some $\gamma \in D_i \setminus R$. Let $h \in H$ be the type of $h_\gamma^{(p_i)}$. Since $h \upharpoonright |R| = 0$ one readily has from (99) that

$$\big|h(v)\big| = \inf_{x\in c_{00}(|R|)} \big|h(v - x)\big| \leqslant \inf_{x\in c_{00}(|R|)} \|v - x\|_{t,H} = d_{t,H}\big(v, c_{00}\big(|R|\big)\big) \leqslant \varepsilon.$$

CASE 2. Suppose that $\|v\|_t = 1$, and that $d_{t,H}(v, c_{00}(|R|)) > \varepsilon \geqslant 1/k$. Because of this last condition, there is $h \in \mathrm{conv}_{\mathbb{Q}}(\pm H)$ such that

$$h \restriction |R| = 0 \quad \text{and} \quad h(x) = \frac{1}{k}. \tag{100}$$

Let $E := \mathrm{Ext}(B_{G^*})$, and let I be an interval of cardinality $\#E$ such that $R < I < D_0$ (this is possible because our assumption on Δ-systems – see Definition 3.24), and let $I = \{\gamma_g\}_{g\in E}$ be an enumeration of it. We define now the following condition $p = (D_p, F_p, A_p, H_p)$:

(a) $D_p = I \cup \bigcup_{i<2k^2n} D_i$ and $A_p = \bigcup_{i<2k^2n} A_i$.
(b) F_p is the minimal symmetric subset of $c_{00}(D_p, \mathbb{Q})$ such that:
  (b.1) It contains $\bigvee_{i<2k^2n} \theta_i(h)$ for every type $h \in F \setminus \pm H$.
  (b.2) For every extremal point g of B_{G^*}, F_p contains the point

$$u_{\gamma_g} \vee \bigvee_{i<n} g(e_i) \cdot \bigvee_{j=ik^2}^{(i+1)k^2-1} \theta_{2j}(h).$$

  (b.3) For every $i < 2k^2n$ and $\gamma \in D_i \setminus R$ one has that F_p contains $h_\gamma^{(p_i)}$.
(c) Given $\gamma \in D_p$, we define

$$h_\gamma^{(p)} := \begin{cases} \bigvee_{j<2k^2n} \theta_j(h) & \text{if } \gamma \in R, \text{ and } h \text{ is the type of } h_\gamma^{(p_0)}, \\ h_\gamma^{(p_i)} & \text{if } \gamma \in D_i \setminus R \text{ for some } i < 2k^2n, \\ u_{\gamma_g} \vee \bigvee_{i<n} g(e_i) \cdot \bigvee_{j=ik^2}^{(i+1)k^2-1} \theta_{2j}(h) & \text{if } \gamma = \gamma_g \text{ for some } g \in E. \end{cases}$$

Then $p \leqslant p_i$ for every $i < 2k^2n$ and $p \in \mathbb{P}$, because of the property (EAMP) of $\mathbb{P}$.

Next, we check that $\mathfrak{P}(v_i)$ holds in p. For each $i < n$, let

$$z_i = \frac{1}{k} \sum_{j<ik^2}^{(i+1)k^2-1} (v_{2j} - v_{2j+1}).$$

We fix scalars $(a_i)_{i<n}$, and set $z = \sum_{i<n} a_i z_i$. We are going to prove first that

$$\|z\|_p = \|z\|_{p,H}: \tag{101}$$

Note that by definition, $F_p \setminus \pm H_p = \{\bigvee_{i<2k^2n} \theta_i(g)\colon\ g \in F\}$, so it follows that for such $f \in F_p \setminus \pm H_p$ one has that $h(z) = 0$, hence we obtain (101).

Now we prove that

$$\|z\|_{p,H} \leqslant \Big\| \sum_{i<n} a_i e_i \Big\|_G:$$

Suppose that $f \in H_p$ is as in (b.3). Then, from (92), one has that

$$\left|f(z)\right| \leqslant \frac{1}{k}\max_{i<n}|a_i| \leqslant \Big\|\sum_{i<n} a_i e_i\Big\|_G.$$

Suppose now that $f = h^{(p)}_{\gamma_g} \in H_p$, $g \in E$, is as in (b.2). Then

$$\begin{aligned} f(z) &= \sum_{i<n} \frac{a_i g(v_i)}{k} \sum_{j=ik^2}^{(i+1)k^2-1} \theta_{2j}(h)(v_{2j}) = \sum_{i<n} a_i g(e_i) = g\Big(\sum_{i<n} a_i e_i\Big) \\ &\leqslant \Big\|\sum_{i<n} a_i e_i\Big\|_G. \end{aligned} \tag{102}$$

Finally, observe that (102) gives the second inequality

$$\Big\|\sum_{i<n} a_i e_i\Big\|_G \leqslant \Big\|\sum_{i<n} a_i z_i\Big\|_p. \qquad \square \tag{103}$$

7.3. Operators on generic spaces

In this section we examine operators on generic spaces over forcing notions with (EAMP). More precisely, we prove the following result.

Theorem 7.8. *Suppose that $\mathbb{P}$ has* (EAMP). *Let $X_{\mathbb{G}}$ be a generic space for $\mathbb{P}$. Let $X \subseteq X_{\mathbb{G}}$ be any of its subspaces. Then every operator $T : X \to X_{\mathbb{G},H}$ is a multiple of the inclusion plus a separable range operator.*

Before we prove this fact, we give some consequences.

Corollary 7.9. *Suppose that $\mathbb{P}$ has* (EAMP). *Every operator $T : X \to X_{\mathbb{G},H}$ from a subspace X of $X_{\mathbb{G},H}$ is a multiple of the inclusion plus a separable range operator.*

Proof. If $\mathbb{P}$ has (EAMP), then so does $\mathbb{P}_H$. Now use that $X_{\mathbb{G},H} = X_{\mathbb{G}_H}$. $\square$

We start the proof of Theorem 7.8 with the following preliminary result. We suppose that $\mathbb{P}$ has (EAMP), and suppose that $\mathbb{G}$ is a generic filter of $\mathbb{P}$.

Lemma 7.10. *Let $(y_\alpha)_{\alpha<\omega_1}$ and $(z_\alpha)_{\alpha<\omega_1}$ be two sequences in $c_{00}(\omega_1,\mathbb{Q}) \cap X_{\mathbb{G}}$ with $(y_\alpha)_\alpha$ normalized and such that*

$$\textit{for every } \alpha<\beta<\omega_1 \textit{ one has that } d_{\mathbb{G}}\big(z_\beta - z_\alpha, \langle y_\beta - y_\alpha\rangle\big) > \varepsilon.$$

Then for every $m \in \mathbb{N}$ there are $\alpha_0 < \cdots < \alpha_{2m-1}$ such that

$$\Big\|\sum_{i<m}(y_{\alpha_{2i+1}} - y_{\alpha_{2i}})\Big\|_{\mathbb{G}} \leqslant 1 \quad \textit{and}$$

$$\Big\|\sum_{i<m}(z_{\alpha_{2i+1}} - z_{\alpha_{2i}})\Big\|_{\mathbb{G}} \geqslant m\varepsilon.$$

Proof. Fix a rational number $\varepsilon > 0$, and $m \in \mathbb{N}$. Let $\mathfrak{P}((v_0, \dots, v_{2m-1}), (w_0, \dots, w_{2m-1}))$ be the following configuration:

(a) Either there is $i < 2m$ such that v_i is not normalized, or there is $i < j < 2m$ such that $d(w_j - w_i, \langle v_j - v_i \rangle) \leqslant \varepsilon$, or
(b) $\| \sum_{i<m} (v_{2i+1} - v_{2i}) \| \leqslant 1$ and $\| \sum_{i<m} (w_{2i+1} - w_{2i}) \| \geqslant m\varepsilon$.

It is clear that the following claim proves the lemma.

Claim 7.10.1. *The metric configuration $\mathfrak{P}((v_i)_{i<2m}, (w_i)_{i<2m})$ is unavoidable for $\mathbb{P}$.*

Proof. We use Theorem 3.36. So, let $(p_i, v_i, w_i)_{i<2m}$ be a Δ-system in $\mathbb{P}$ of type $t = (N, F, A, H, v, w)$ and with root R. For each $i < 2m$, let $\theta_i : N \to D_i$ be the corresponding order-preserving bijection. There are three cases to consider:

CASE 1. Suppose first that $\|v\|_t \neq 1$. Then any amalgamation of $(p_i)_{i<2m}$ will satisfy the configuration $\mathfrak{P}$.

CASE 2. Now suppose that

$$d_{t,H}\big(w, \big\langle c_{00}\big(|R|\big) \cup \{v\}\big\rangle\big) \leqslant \varepsilon.$$

Let $p = (\bigcup_{i<2m} D_{p_i}, F_p, A_p, H_p)$ be the basic amalgamation of $(p_i)_{i<2m}$. Then $p \in \mathbb{P}$ because $\mathbb{P}$ has the (EAMP). We check that $\mathfrak{P}$ holds in X_p. Let $a \in \mathbb{R}$ and $n \in c_{00}(R)$ be such that $\|w - av + n\|_t \leqslant \varepsilon$. Then for every $i < j < 2m$ one has that

$$\big\| w_j - w_i - a(v_j - v_i) \big\|_p \leqslant \varepsilon: \tag{104}$$

If $f = \bigvee_{i<2m} \theta_i(g)$ with $g \in F$, then $f(w_j - w_i - a(v_j - v_i)) = g(w) - g(w) - a(g(v) - g(v)) = 0$. Suppose that $f = h_\gamma^{(p_k)}$ for $k < 2m$ and $\gamma \in D_i \setminus R$. Let $g \in F$ be the type of $h_\gamma^{(p_i)}$. If $k \neq i, j$ then $f(w_j - w_i - a(v_j - v_i)) = 0$. Suppose that $k = i$ or $k = j$. Then

$$\begin{aligned} \big| f\big(w_j - w_i - a(v_j - v_i)\big)\big| &= \big| h_\gamma^{(p_k)}(w_k - a v_k)\big| = \big| h(w - av - n)\big| \\ &\leqslant \|w - av - n\|_{t,H} \leqslant \varepsilon. \end{aligned} \tag{105}$$

CASE 3. Suppose that $\|v\|_t = 1$, and that

$$d_{t,H}\big(w, \big\langle c_{00}\big(|R|\big) \cup \{v\}\big\rangle\big) > \varepsilon. \tag{106}$$

Let $h \in \mathrm{conv}_{\mathbb{Q}}(\pm H)$ be such that

$$h \restriction \big\langle c_{00}\big(|R|\big) \cup \{v\}\big\rangle = 0 \quad \text{and} \quad h(w) = \varepsilon.$$

Note that h exists because of the assumption (106). Let $R < \bar{\gamma} < D_0 \setminus R$, and let $p = (D_p, F_p, A_p, H_p)$ be the following condition: $D_p = \{\bar{\gamma}\} \cup \bigcup_{i<2m} D_i$ and $A_p = \bigcup_{i<2m} A_i$. F_p is the minimal symmetric subset of $c_{00}(D_p, \mathbb{Q})$ containing

(a) $\bigvee_{i<2m} \theta_i(g)$ for every type $g \in F \setminus \pm H$.
(b) $u_{\bar{\gamma}} \vee \bigvee_{i<m} \theta_{2i+1}(h)$.
(c) $h_\gamma^{(p_i)}$ for every $i < 2m$ and $\gamma \in D_i \setminus R$.

Given $\gamma \in D_p$, we define

$$h_\gamma^{(p)} := \begin{cases} \bigvee_{j<2m} \theta_j(h) & \text{if } \gamma \in R, \text{ and } h \text{ is the type of } h_\gamma^{(p_0)}, \\ h_\gamma^{(p_i)} & \text{if } \gamma \in D_i \setminus R \text{ for some } i < 2m, \\ u_{\bar{\gamma}} \vee \bigvee_{i<m} \theta_{2i+1}(h) & \text{if } \gamma = \bar{\gamma}. \end{cases}$$

Then $p \leqslant p_i$ for every $i < 2m$ and $p \in \mathbb{P}$, because of the property (EAMP) of $\mathbb{P}$.

We check that $\mathfrak{P}(v_i)_{i<2m}$ holds in p. It is routine to check that

$$\Big\| \sum_{i<m} (v_{2i+1} - v_{2i}) \Big\|_p \leqslant 1.$$

On the other hand,

$$\Big\| \sum_{i<m} (w_{2i+1} - w_{2i}) \Big\|_p \geqslant \Big(\bigvee_{i<m} \theta_{2i+1}(h) \Big) \Big(\sum_{i<m} w_{2i+1} - w_{2i} \Big) = m \cdot h(w) = m\varepsilon. \qquad \square$$

The proof of Theorem 7.8 is based on the following finer result.

Lemma 7.11. *Let $(x_\alpha)_{\alpha<\omega_1}$ be a normalized separated sequence in $X_{\mathbb{G}}$, and let X be its closed linear span. Then for every bounded operator $T : X \to X_{\mathbb{G}}$ there are $\bar{\alpha} < \omega_1$ and $\lambda \in \mathbb{R}$ such that for every $\alpha \geqslant \bar{\alpha}$ one has that*

$$T(x_\alpha) - \lambda x_\alpha \in X_{\mathbb{G}}^{(\bar{\alpha})}. \tag{107}$$

Proof. Fix all given data.

Claim 7.11.1. *There is $\alpha_0 < \omega_1$ such that for every $\alpha \geqslant \alpha_0$ one has that*

$$T(x_\alpha) \in \overline{\big\langle X_{\mathbb{G}}^{(\alpha_0)} \cup \{x_\alpha\} \big\rangle}. \tag{108}$$

Proof. Working towards a contradiction, suppose that such $\alpha_0 < \omega_1$ does not exist.

Then, using that $(x_\alpha)_{\alpha<\omega_1}$ is a separated sequence, it is not difficult to find an uncountable subsequence $(x_{\alpha_\xi})_{\xi<\omega_1}$ such that for every $\xi < \omega_1$ one has that

$$x_{\alpha_\xi} \notin \overline{\big\langle T(x_{\alpha_\eta}) \big\rangle_{\eta<\xi} + \langle x_{\alpha_\eta} \rangle_{\eta<\xi}}, \quad \text{and}$$
$$T(x_{\alpha_\xi}) \notin \overline{\big\langle T(x_{\alpha_\eta}) \big\rangle_{\eta<\xi} + \langle x_{\alpha_\eta} \rangle_{\eta\leqslant\xi}}.$$

By going to a further uncountable subsequence and after re-enumeration if needed, we assume that there is $0 < \varepsilon < 1$ such that for every $\alpha < \omega_1$ one has that

$$d\big(x_\alpha, \langle x_\beta\rangle_{\beta<\alpha} + \big\langle T(x_\beta)\big\rangle_{\beta<\alpha}\big) \geqslant \varepsilon, \quad \text{and}$$
$$d\big(T(x_\alpha), \big\langle T(x_\beta)\big\rangle_{\beta<\alpha} + \langle x_\beta\rangle_{\beta\leqslant\alpha}\big) \geqslant \varepsilon.$$

Using this, the fact that T is bounded and a simple approximation argument, we can find $\delta_0 > 0$ such that if $(y_\alpha)_{\alpha<\omega_1}$ and $(z_\alpha)_{\alpha<\omega_1}$ are sequences in $c_{00}(\omega_1, \mathbb{Q})$ such that

$$\sup_{\alpha<\omega_1} \big\{\|x_\alpha - y_\alpha\|, \big\|T(x_\alpha) - z_\alpha\big\|\big\} \leqslant \delta_0 \tag{109}$$

then for every $\alpha < \beta < \omega_1$ one has that $d(z_\beta - z_\alpha, \langle y_\beta - y_\alpha\rangle) \geqslant \varepsilon/2$. Let $m \in \mathbb{N}$ be such that

$$m\frac{\varepsilon}{2} > 2\|T\| + 1. \tag{110}$$

Let now $(y_\alpha)_{\alpha<\omega_1}$ and $(z_\alpha)_{\alpha<\omega_1}$ in $c_{00}(\omega_1, \mathbb{Q})$ be such that

$$\|x_\alpha - y_\alpha\|, \big\|T(x_\alpha) - z_\alpha\big\| \leqslant \min\left\{\delta_0, \frac{1}{2m}\right\} \quad \text{for every } \alpha < \omega_1. \tag{111}$$

Since x_α is normalized, we may assume without loss of generality that y_α is also normalized. By Lemma 7.10, there are $\{\alpha_i\}_{i<2m}$ such that

$$\bigg\|\sum_{i<m} y_{\alpha_{2i+1}} - y_{\alpha_{2i}}\bigg\| \leqslant 1, \tag{112}$$

$$\bigg\|\sum_{i<m} z_{\alpha_{2i+1}} - z_{\alpha_{2i}}\bigg\| \geqslant m\frac{\varepsilon}{2} > 2\|T\| + 1. \tag{113}$$

Let $x = \sum_{i<m} x_{\alpha_{2i+1}} - x_{\alpha_{2i}}$. Then

$$\|x\| \leqslant \bigg\|\sum_{i<m} y_{\alpha_{2i+1}} - y_{\alpha_{2i}}\bigg\| + \sum_{i<2m} \|x_{\alpha_i} - y_{\alpha_i}\| \leqslant 2, \tag{114}$$

while

$$\big\|T(x)\big\| \geqslant \bigg\|\sum_{i<m} z_{\alpha_{2i+1}} - z_{\alpha_{2i}}\bigg\| - \sum_{i<2m} \big\|T(x_{\alpha_i}) - z_{\alpha_i}\big\| > 2\|T\| + 1 - 1 > \|T\|\|x\| \tag{115}$$

which is impossible since T is bounded. □

Using Claim 7.11.1 we fix some $\alpha_0 < \omega_1$ and for each $\alpha \geqslant \alpha_0$ a scalar $\lambda_\alpha \in \mathbb{R}$ such that

$$T(x_\alpha) - \lambda_\alpha^{(T)} x_\alpha \in X_{\alpha_0}. \tag{116}$$

Claim 7.11.2. *There is some $\alpha_0 \leqslant \bar{\alpha} < \omega_1$ such that $\lambda_\alpha = \lambda_\beta$ for all $\bar{\alpha} \leqslant \alpha, \beta < \omega_1$.*

It is clear that this claim proves the lemma.

Proof of Claim 7.11.2. Suppose otherwise, and find a sequence $(\beta_\xi, \delta_\xi)_{\xi<\omega_1}$ such that:

(a) $\beta_\xi < \delta_\xi < \beta_{\xi+1}$ for every $\xi < \omega_1$.
(b) $\lambda_{\beta_\xi} \neq \lambda_{\delta_\xi}$ for every $\xi < \omega_1$.
(c) $x_{\beta_\xi} \notin X_{\mathbb{G}}^{(\xi)}$ and $x_{\delta_\xi} \notin \overline{\langle X_{\mathbb{G}}^{(\xi)} \cup \{x_{\beta_\xi}\}\rangle}$ for every $\xi < \omega_1$.

Let $Y \subseteq X$ be the closed linear span of $(z_\xi)_{\xi<\omega_1}$, where $z_\xi = (x_{\delta_\xi} - x_{\beta_\xi})$, for every $\xi < \omega_1$, and $T_0 = T \restriction Y$. It is clear that $(z_\xi/\|z_\xi\|)_{\xi<\omega_1}$ is a separated normalized sequence. By applying Claim 7.11.1 to it, we obtain that there is some $\alpha_0 \leqslant \alpha_1 < \omega_1$, and for each $\xi \geqslant \alpha_1$ a scalar η_ξ such that

$$T_0(z_\xi) - \eta_\xi z_\xi \in X_{\mathbb{G}}^{(\alpha_1)}. \tag{117}$$

In particular, if we take $\xi = \alpha_1$, it follows from (116) and (117) that

$$\eta_{\alpha_1}(x_{\delta_{\alpha_1}} - x_{\beta_{\alpha_1}}) - (\lambda_{\delta_{\alpha_1}} x_{\delta_{\alpha_1}} - \lambda_{\beta_{\alpha_1}} x_{\beta_{\alpha_1}}) \in X_{\mathbb{G}}^{(\alpha_1)}. \tag{118}$$

Hence

$$(\eta_{\alpha_1} - \lambda_{\delta_{\alpha_1}}) x_{\delta_{\alpha_1}} \in \big\langle X_{\mathbb{G}}^{(\alpha_1)} \cup \{x_{\beta_{\alpha_1}}\}\big\rangle. \tag{119}$$

It follows from (c) that

$$\eta_{\alpha_1} = \lambda_{\delta_{\alpha_1}}.$$

Hence, from this and (118), we obtain that

$$(\eta_{\alpha_1} - \lambda_{\beta_{\alpha_1}}) x_{\beta_{\alpha_1}} \in X_{\mathbb{G}}^{(\alpha_1)}. \tag{120}$$

Again using (c), it follows that

$$\lambda_{\beta_{\alpha_1}} = \eta_{\alpha_1} = \lambda_{\delta_{\alpha_1}},$$

which is contradictory with (b). □

Proof of Theorem 7.8. Fix all given data. Let $(x_\alpha)_{\alpha<\omega_1}$ be a fundamental and separated normalized sequence of X. Let $\bar{\alpha} < \omega_1$ and $\lambda \in \mathbb{R}$ be given by Lemma 7.11 applied to T and that separated sequence. Let us prove that $U = T - \lambda i_{X, X_{\mathbb{G}}}$ has separable range, indeed $\mathrm{Im}(U) \subseteq X_{\mathbb{G}}^{(\bar{\bar{\alpha}})}$ where $\bar{\alpha} \leqslant \bar{\bar{\alpha}} < \omega_1$ is such that

$$T\big(\overline{\langle x_\alpha\rangle_{\alpha<\bar{\alpha}}}\big) \subseteq X_{\mathbb{G}}^{(\bar{\bar{\alpha}})}.$$

So, fix $x \in X$. Let $\varepsilon > 0$, and let $y \in \langle x_\alpha\rangle_{\alpha<\omega_1}$ be such that $\|y - x\| \leqslant \varepsilon$. Let also $v \in \langle y_\alpha\rangle_{\alpha<\bar{\alpha}}$ and $w \in \langle y_\alpha\rangle_{\alpha\geqslant\bar{\alpha}}$ be such that $y = v + w$. It follows that

$$U(y) = U(v) + U(w) = U(v) + \lambda w + \bar{w} - \lambda w = U(v) + \bar{w},$$

where $\bar{w} \in X_{\mathbb{G}}^{(\bar{\alpha})}$. So,

$$U(y) \in X_{\mathbb{G}}^{(\bar{\bar{\alpha}})}. \tag{121}$$

Since

$$\|U(x) - U(y)\| \leqslant \|U\| \|x - y\| = \|U\| \varepsilon \tag{122}$$

and $\varepsilon > 0$ is arbitrary, it follows from (121) and (122) that $U(x) \in X_{\mathbb{G}}^{(\bar{\bar{\alpha}})}$, as desired. □

Corollary 7.12. *Suppose that $\mathbb{P}$ has the* (EAMP). *Then any generic space $X_{\mathbb{G}}$ of $\mathbb{P}$ is ω_1-hereditarily indecomposable, i.e. the distance between two non-separable subspaces of $X_{\mathbb{G}}$ is zero.*

Proof. This is a standard property of the Banach spaces having few operators as in Theorem 7.8. We reproduce here its proof. Fix two non-separable subspaces X and Y of $X_{\mathbb{G}}$. Using that both are non-separable, find 1/2-separated and normalized sequences $(x_\alpha)_{\alpha<\omega_1}$ and $(y_\alpha)_{\alpha<\omega_1}$ in X and Y respectively such that in addition $d(y_\beta, \langle \{x_\alpha\}_{\alpha\leqslant\beta} \cup \{y_\alpha\}_{\alpha<\beta}\rangle) \geqslant 1/2$ for every $\beta < \omega_1$. It follows from this that $(y_\alpha - x_\alpha)_{\alpha<\omega_1}$ is a 1/2-separated seminormalized sequence. Let Z be the linear span of $(y_\alpha - x_\alpha)_\alpha$, and define linearly $T_X : Z \to \langle x_\alpha\rangle_{\alpha<\omega_1}$ and $T_Y : Z \to \langle y_\alpha\rangle_{\alpha<\omega_1}$ by $T_X(y_\alpha - x_\alpha) = x_\alpha$, $T_Y(y_\alpha - x_\alpha) = y_\alpha$. There are two cases to consider:

CASE 1. Either T_X or T_Y is not bounded. Suppose without loss of generality that T_X is not bounded. Then for every $\varepsilon > 0$ there is some finite sequence $(a_\alpha)_{\alpha\in s}$ of scalars, $s \subseteq \omega_1$, such that $\|\sum_{\alpha\in s} a_\alpha y_\alpha - \sum_{\alpha\in s} a_\alpha x_\alpha\| \leqslant \varepsilon$ and $\|\sum_{\alpha\in s} a_\alpha x_\alpha\| = 1$. Hence, it follows that $d(S_X, S_Y) = 0$.

CASE 2. Both T_X and T_Y are bounded operators. Let U_X and U_Y be their extensions to the closure $\overline{Z}$ of Z. By Theorem 7.8 applied to U_X there is a scalar λ such that $S = U_X - \lambda i_{\overline{Z}, X_{\mathbb{G}}}$ has separable range. Obviously, U_X does not have separable range itself. Hence $\lambda \neq 0$. If there was the case that $\lambda = 1$ then using that $U_Y = i_{\overline{Z}, X_{\mathbb{G}}} + U_X$, it follows that $U_Y = (\lambda + 1) i_{\overline{Z}, X_{\mathbb{G}}} + S$, and $\lambda + 1 = 2 \neq 1$. So by replacing U_X with U_Y if needed, we may assume, without loss of generality, that $\lambda \neq 1$.

Now for every $\varepsilon > 0$ there is some normalized $z \in Z$, $z = \sum_{\alpha\in s} a_\alpha(y_\alpha - x_\alpha)$ such that

$$\left\| \sum_{\alpha\in s} a_\alpha x_\alpha - \lambda z \right\| \leqslant \varepsilon. \tag{123}$$

This means that

$$\left\| (1 - \lambda) \sum_{\alpha\in s} a_\alpha x_\alpha - \sum_{\alpha\in s} a_\alpha y_\alpha \right\| \leqslant \varepsilon. \tag{124}$$

In particular, if we take $\varepsilon < \lambda/2$, then (123) gives that $\|\sum_{\alpha\in s} a_\alpha x_\alpha\| \geqslant \lambda/2$, and hence

$$\left\| (1 - \lambda) \sum_{\alpha\in s} a_\alpha x_\alpha \right\| \geqslant \frac{1}{2} |1 - \lambda| \lambda. \tag{125}$$

Since $0 < \varepsilon < \lambda/2$ is arbitrary, it is now easy to see that (124) and (125) imply that $d(S_X, S_Y) = 0$, as desired. □

7.4. Quotients of generic spaces

We finish this section by analyzing the quotients of generic spaces. We prove in particular that most of our generic spaces have separable quotients.

Theorem 7.13. *Suppose that $\mathbb{P}$ has the property* (SAMP). *Let $X_{\mathbb{G}}$ be any $\mathbb{P}$-generic space. Let $T : X_{\mathbb{G}} \to X$ be any quotient map. Then either* $\mathrm{Ker}(T)$ *or X is separable.*

Remark 7.14. We note that if the space $X_{\mathbb{G}}$ has an uncountable biorthogonal system $(x_\alpha)_{\alpha<\omega_1}$ and if we let X be the closed linear span of $(x_{2\alpha})_{\alpha<\omega_1}$ then both X and $X_{\mathbb{G}}/X$ are non-separable, so the assumption of the property (SAMP) for $\mathbb{P}$ looks somehow necessary since it goes opposite to the existence of uncountable biorthogonal systems.

Proof. Working towards a contradiction, suppose that $T : X_{\mathbb{G}} \to X$ is a quotient map such that $\mathrm{Ker}(T)$ and X are not separable. Then we can find two $3/4$-separated and normalized sequences $(x_\alpha)_{\alpha<\omega_1}$ and $(y_\alpha)_{\alpha<\omega_1}$ such that $x_\alpha \in X$ and such that

$$d\big(y_\beta, \langle X \cup \{y_\alpha\}\rangle\big) \geqslant \frac{3}{4} \quad \text{for every } \alpha < \beta < \omega_1. \tag{126}$$

We proceed as in the proof of Theorem 7.3. Fix $(\varepsilon_a)_{a>0}$ witnessing that $\mathbb{P}$ has the property (SAMP). We use Proposition 3.7(e) to the sequence $(y_\alpha)_{\alpha<\omega_1}$ and $\varepsilon = 1/4$ to find $\bar{\delta} < 1/4$ and an uncountable $\Gamma \subseteq \omega_1$ such that

$$\text{if } f \in B_{(X_{\mathbb{G}})^*} \text{ is such that } \|f\|_\infty \leqslant \bar{\delta}, \text{ then } \big|f(y_\alpha)\big| \leqslant 1/4 \text{ for every } \gamma \in \Gamma. \tag{127}$$

After re-enumeration if needed, we assume that $\Gamma = \omega_1$. Let $\bar{m}$ be the first integer m such that

$$\bar{m} \cdot \varepsilon_{\bar{\delta}} \geqslant 1. \tag{128}$$

Let now $(v_\alpha)_{\alpha<\omega_1}$ and $(w_\alpha)_{\alpha<\omega_1}$ be two sequences of normalized vectors of $c_{00}(\omega_1, \mathbb{Q})$ such that

$$\|x_\alpha - v_\alpha\|, \|y_\alpha - w_\alpha\| \leqslant \min\left\{\bar{\delta}, \frac{1}{8(2\bar{m}+3)}\right\} \quad \text{for all } \alpha < \omega_1. \tag{129}$$

It follows then from (126) and (129) that for every $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_{2\bar{m}+2}$ one has that

$$\left\| -\frac{1}{2}(w_{\alpha_0} + w_{\alpha_1}) + w_{\alpha_2} - \frac{1}{2}\sum_{i=1}^{\bar{m}} (v_{\alpha_{2i+2}} - v_{\alpha_{2i+1}}) \right\| > \frac{1}{2}. \tag{130}$$

Claim 7.14.1. *Let $\mathfrak{P}(v_0, \dots, v_{2\bar{m}+2}, w_0, \dots, w_{2\bar{m}+2})$ be the following configuration*:

(I) *Either there is some $i < 2\bar{m}+3$ such that either v_i or w_i is not normalized, or there is some $f \in B_{X^*}$ with $\|f\|_\infty \leqslant \bar{\delta}$ and some $i < 2\bar{m}+3$ such that $|f(w_i)| > 1/2$, or*
(II) *there is some $i < j < 2\bar{m}+3$ such that $\|v_i - v_j\| \leqslant 1/2$, or*

(III) *neither* (I) *and* (II) *holds and*

$$\left\| -\frac{1}{2}(w_0 + w_1) + w_2 - \frac{1}{\bar{m}} \sum_{i=1}^{\bar{m}} (v_{2i+2} - v_{2i+1}) \right\| \leqslant \frac{1}{2}. \tag{131}$$

Then $\mathfrak{P}(v_0, \ldots, v_{2\bar{m}+2}, w_0, \ldots, w_{2\bar{m}+2})$ *is unavoidable.*

Since (I) and (II) above are not unavoidable for the sequences $(v_\alpha)_{\alpha<\omega_1}$ and $(w_\alpha)_{\alpha<\omega_1}$, it follows from the fact that $\mathfrak{P}$ is unavoidable that there are $\alpha_0 < \cdots < \alpha_{2\bar{m}+2}$ such that (130) does not hold, a contradiction. It rests to prove the claim.

Proof of Claim 7.14.1. We use the Forcing Theorem. Let $(p_i, v_i, w_i)_{i<2\bar{m}+3}$ be a Δ-system of type $t = (N, F, A, H, v, w)$ and root R. Let $\theta_i : N \to D_i$ be the order-preserving bijection for every $i < 2\bar{m} + 3$.

CASE 1. Suppose that

(∗) either v or w is not normalized in X_t, or if there is some $f \in B_{(X_t)^*}$ such that $\|f\|_\infty \leqslant \bar{\delta}$ and such that $|f(w)| > 1/2$,

then any amalgamation (for example the basic one) of $(p_i)_i$ will have the property (I).

CASE 2. Suppose that (∗) does not hold.

CASE 2.1 Suppose that

(∗∗) $d_{X_{t,H}}(v, c_{00}(|R|)) < \frac{1}{2}$.

Let p be the basic amalgamation of $(p_i)_{i<2\bar{m}+3}$. Then it is easy to see that

$$\|v_4 - v_5\|_p = \max\bigl\{\bigl|h_\gamma^{(p_i)}\bigr|\colon\ i = 0, 1 \text{ and } \gamma \in D_i \setminus R\bigr\}. \tag{132}$$

Since for $i = 4, 5$ and $\gamma \in D_i \setminus R$ one has that

$$\bigl|h_\gamma^{(p_i)}(v_4 - v_5)\bigr| = \bigl|h_\gamma^{(p_i)}(v_i)\bigr| = \inf_{x \in c_{00}(R)} \bigl|h_\gamma^{(p_i)}(v_i + v)\bigr| \leqslant d_{X_{t,H}}\bigl(v, c_{00}\bigl(|R|\bigr)\bigr) < \frac{1}{2}, \tag{133}$$

it follows from (132) and (133) that

$$\|v_4 - v_5\|_p < \frac{1}{2}, \tag{134}$$

so (II) holds in X_p.

CASE 2.2 Suppose that $d_{X_{t,H}}(v, c_{00}(|R|)) > 1/2$. We find $h \in \mathrm{conv}_{\mathbb{Q}}(\pm H_t)$ such that

$$h \restriction c_{00}\bigl(|R|\bigr) = 0 \quad \text{and} \quad h(v) = \frac{1}{2}. \tag{135}$$

Define the condition $p = (\bigcup_{i<2\bar{m}+3} D_i, F_p, \bigcup_{i<2\bar{m}+3} A_i, H_p)$ as follows. F_p is the minimal symmetric subset of $c_{00}(D_p)$ such that

(a) $\bigvee_{i<2\bar{m}+3}\theta_i(g)$ is in F_p for every type $g\in F$.
(b) $h_\gamma^{(p_i)}$ is in F_p for every $i<2\bar{m}+3$, $i\neq 2$ and every $\gamma\in D_i$, or for $i=2$ and $\gamma\in A_i^{(\delta)}$ with $(\delta)_\infty<\bar{\delta}$,
(c) F_p contains

$$h_\gamma^{(p_2)}\vee\frac{h_\gamma^{(p_2)}(w)}{\bar{m}}\cdot\bigvee_{i=3}^{2\bar{m}+2}(-1)^i\theta_i(h)$$

for every $\gamma\in D_2\setminus(A_2\cup R)$, and every $\gamma\in A_2^{(\gamma)}\setminus R$ with $(\delta)_\infty\geqslant\bar{\delta}$.

Fix $i<2\bar{m}+2$, $\gamma\in D_i$, let $g\in F$ be the type of $h_\gamma^{(p_i)}$, and set $\delta:=(h_\gamma^{(p_i)})_\gamma$. Define

$$h_\gamma^{(p)}:=\begin{cases}\bigvee_{j<2m}\theta_j(g) & \text{if } \gamma\in R,\\ h_\gamma^{(p_i)} & \text{if } R<\gamma\text{, and } i\neq 2 \text{ or if } (\delta)_\infty<\bar{\delta},\\ h_\gamma^{(p_2)}\vee\frac{h_\gamma^{(p_2)}(w)}{\bar{m}}\cdot\bigvee_{i=3}^{2\bar{m}+2}(-1)^i\theta_i(h) & \text{if } R<\gamma,\ i=2 \text{ and } n\leqslant\bar{n}.\end{cases}$$

Since $\mathbb{P}$ has the property (SAMP), it follows by the choice of $\bar{m}$ that $p\in\mathbb{P}$. The proof will be finished once we justify that (131) holds in X_p: Let

$$z=-\frac{1}{2}(w_0+w_1)+w_2-\frac{1}{2}\sum_{i=1}^{\bar{m}}(v_{2i+1}-v_{2i}),$$

and fix $f\in F_p$. Suppose first that f is as in (a), i.e. $f=\bigvee_{i<2\bar{m}+2}\theta_i(g)$ for some type $g\in F$. Then

$$f(w)=-\frac{1}{2}\big(g(w)+g(w)\big)+g(w)-\frac{1}{2}\sum_{i=1}^{\bar{m}}\big(g(v)-g(v)\big)=0. \tag{136}$$

Suppose that f is as in (b), i.e. $f=h_\gamma^{(p_i)}$ for some $i<2\bar{m}+3$, with $R<\gamma$ and such that either $i\neq 2$ or $\gamma\in A_i^{(\delta)}$ with $(\delta)_\infty<\bar{\delta}$. If it is the case that $i\neq 2$, then

$$\big|f(z)\big|=\frac{1}{2}\big|h_\gamma^{(p_i)}(x)\big|\leqslant\frac{1}{2}, \tag{137}$$

where $x=w$ if $i=0,1$ and $x=v$ otherwise. If $i=2$ and $(\delta)_\infty<\bar{\delta}$, then by the negation of $(*)$, it follows that

$$\big|f(z)\big|=\frac{1}{2}\big|h_\gamma^{(p_2)}(w_2)\big|\leqslant\frac{1}{2}. \tag{138}$$

Finally, suppose that f is as in (c), i.e.,

$$f = h_\gamma^{(p_2)} \vee \frac{1}{\bar{m}} \bigvee_{i=3}^{2\bar{m}+2} (-1)^{i+1}\theta_i(h),$$

where $\gamma \in A_2^{(\delta)} \setminus R$ for some δ such that $(\delta)_\infty \geqslant \bar{\delta}$. Then

$$\begin{aligned} |f(z)| &= \left| h_\gamma^{(p_2)}(w) - \frac{h_\gamma^{(p_2)}(w)}{\bar{m}} \sum_{i=1}^{\bar{m}} \big(h(v) + h(v)\big) \right| \\ &= \left| h_\gamma^{(p_2)}(w) - \frac{h_\gamma^{(p_2)}(w)}{\bar{m}} \bar{m} \right| = 0. \qquad \square \end{aligned} \tag{139}$$

8. Generic Choquet simplexes

In this section we examine dual balls (or their convex pieces) of the generic Banach spaces considered above. In fact, we are going to reformulate the forcing construction avoiding the direct reference to the Banach space. It turns out that these generic constructions will provide not only compact convex sets but more specific objects in the following sense.

Definition 8.1. A convex subset K of a linear space X is a *Choquet simplex* whenever the associated cone $C = \{(\lambda x, \lambda)\colon\ x \in K,\ \lambda \geqslant 0\}$ defines a lattice order on the set $C - C \subseteq X \times \mathbb{R}$.

We refer the reader to [9] and [32] for more information about this notion. Note that in the case of a convex set $K \subseteq \mathbb{R}^n$, K is a simplex if and only if $\#(\mathrm{Ext}(K)) = n + 1$. We shall base our constructions of generic Choquet simplexes on the following standard fact.

Proposition 8.2. *Let $(D_i)_{i\in I}$ be a directed family of finite sets, and for each $i \in I$, let $K_i \subseteq \mathbb{R}^{D_i}$ be a simplex such that for $i \leqslant j$ one has that $\pi_{i,j}(K_j) = K_i$, where $\pi_{i,j} : \mathbb{R}^{D_j} \to \mathbb{R}^{D_i}$ is the natural restriction. Then the compact and convex set*

$$K = \bigcap_{i\in I} \pi_i^{-1}(K_i) \tag{140}$$

is a simplex, where $\pi_i : \mathbb{R}^D \to \mathbb{R}^{D_i}$ is the corresponding projection for $D = \bigcup_{i\in I} D_i$.

Proof. Use the characterization of a simplex exposed in [9, Theorem 3.2, p. 613]. $\square$

We present now some constructions of generic compacta and generic simplexes K. They will be closed subsets of some Tychonoff cube of the form $[-1, 1]^I$. Since we are interested in properties of the corresponding Banach space $C(K)$ of real-valued continuous functions on K, it is useful to introduce the notion of polynomial on $C([-1, 1]^A)$. Given $\gamma \in A$, let $\pi_\gamma : [-1, 1]^A \to [-1, 1]$ be defined by $\pi_\gamma(f) = (f)_\gamma$. We say that $p \in C([-1, 1]^A)$ is a *polynomial* if it is in the algebra generated by $\{\pi_\gamma\}_{\gamma\in A}$ and the constant functions on $[-1, 1]^A$. It is a consequence of the Stone–Weierstrass Theorem that the polynomials are dense in $C([-1, 1]^A)$. Many of the properties of our generic simplexes $K_{\mathbb{G}}$ will be obtained by examining the corresponding space $P(K_{\mathbb{G}})$ of Radon probability measures on $K_{\mathbb{G}}$ with its natural weak* topology when considered as a subset of the dual of the function space $C(K_{\mathbb{G}})$.

The constructions of the generic simplexes and compacta will be in some sense dual to the constructions of generic Banach spaces presented above. In particular, the important Theorem 3.36 is now true for uncountable sequences of polynomials with rational coefficients (Forcing Theorem in this context).

Definition 8.3. The basic forcing $\mathbb{T}$ for introducing generic compacta is the following: The conditions are $p=(D_p, F_p, H_p)$ such that

(K.0) $D_p \subseteq \omega_1$ is finite.
(K.1) $H_p \subseteq F_p \subseteq c_{00}(D_p)$ are finite and F_p has the property that $F_p \cap -F_p=\{0\}$.
(K.2) $H_p=\{h_\gamma^{(p)}\}_{\gamma\in D_p}$ is such that for every $\gamma\in D_p$ one has that $(h_\gamma^{(p)})_\gamma=1$ and $h_\gamma^{(p)} \upharpoonright \gamma=0$.

The ordering $p\leqslant q$ for $p,q\in\mathbb{T}$ is defined by

(O.1) $D_q\subseteq D_p$.
(O.2) $F_q\subseteq F_p\upharpoonright D_q\subseteq \mathrm{conv}_{\mathbb{Q}}(F_q)$.
(O.3) $H_q\subseteq H_p\upharpoonright D_q\subseteq H_q\cup\{0\}$.

The main differences between this partial ordering and the basic forcing notion is that F_p is not symmetric (indeed it is asymmetric), and that in the extension, $p\leqslant q$, if $\gamma\in D_p\setminus D_q$ then $h_\gamma^{(p)}\upharpoonright D_q$ is either equal to some $h_\eta^{(0)}$ with $\eta\in D_q$, or equal to 0.

The following is easy to prove.

Proposition 8.4. *The forcing notion $\mathbb{T}$ has* (AMP)*, the* (EP) *and the Shanin property.* □

Definition 8.5. Fix a generic filter $\mathbb{G}$ for $\mathbb{T}$. Then we can define the corresponding generic compacta

$$K_{\mathbb{G}} := \bigcap_{p\in\mathbb{G}} \pi_{D_p}^{-1}\big(\mathrm{conv}_{\mathbb{R}}(F_p)\big),$$

$$L_{\mathbb{G}} := \bigcap_{p\in\mathbb{G}} \pi_{D_p}^{-1}\big(\mathrm{conv}_{\mathbb{R}}\big(H_p\cup\{0\}\big)\big), \quad \text{and}$$

$$H_{\mathbb{G}} := \big\{h_\gamma^{(\mathbb{G})}\colon \gamma<\omega_1\big\}\cup\{0\}.$$

It is clear that $K_{\mathbb{G}}$ and $L_{\mathbb{G}}$ are both compact convex subsets of $\mathbb{R}^{\omega_1}$ and that $H_{\mathbb{G}}\subseteq L_{\mathbb{G}}\subseteq K_{\mathbb{G}}$.

Proposition 8.6. *The compactum $H_{\mathbb{G}}$ is a non-metrizable scattered compactum. It follows that $L_{\mathbb{G}}$ and $K_{\mathbb{G}}$ are non-metrizable compacta as well.*

Proof. To prove that $H_{\mathbb{G}}$ is compact we see that

$$H_{\mathbb{G}} = \bigcap_{p\in\mathbb{G}} \pi_{D_p}^{-1}\big(H_p\cup\{0\}\big): \tag{141}$$

The direct inclusion: Let $\gamma<\omega_1$, and fix $p\in\mathbb{G}$. Let $q\in\mathbb{G}$ with $q\leqslant p$ and $\gamma\in D_q$. Then it follows by condition (O.3) on the extension $\leqslant$ that

$$h_\gamma^{(\mathbb{G})} \restriction D_p = h_\gamma^{(q)} \restriction D_p \in H_p \cup \{0\}, \tag{142}$$

so we are done. Now suppose that f is in the right-hand set in (141), and suppose that $f \neq 0$. Let γ be the minimal ordinal such that $(f)_\gamma \neq 0$. We claim that then $f = h_\gamma^{(\mathbb{G})}$: Suppose that this is not the case, and let now $p \in \mathbb{G}$ be such that $\gamma \in D_p$ and $f \restriction D_p \neq h_\gamma^{(p)}$. Fix $\eta \in D_p \setminus \{\gamma\}$ such that $f \restriction D_p = h_\eta^{(p)}$. If $\gamma < \eta$, then $1 = (h)_\gamma = (h_\eta^{(p)})_\gamma$, which is impossible by definition of $\mathbb{T}$. Otherwise, $\eta < \gamma$, and then $1 = (h_\eta^{(p)})_\eta = (h)_\eta$, contradicting the fact that γ is the minimal element of the support of f.

We verify that $H_{\mathbb{G}}$ is scattered: Fix $C \subseteq H_{\mathbb{G}}$ is closed and nonempty, suppose that $C \neq \{0\}$. Let

$$\gamma = \min\{\eta\colon\ h_\eta^{(\mathbb{G})} \in C\}.$$

Then $h_\gamma^{(\mathbb{G})}$ is isolated in C: The open neighborhood $U := \{f \in H_{\mathbb{G}}\colon\ (f)_\gamma > 1/2\}$ of $h_\gamma^{(\mathbb{G})}$ is such that $U \cap C = \{h_\gamma^{(\mathbb{G})}\}$.

We check now that $H_{\mathbb{G}}$ is non-metrizable. For this, it suffices to prove that the sequence $(\pi_\gamma)_{\gamma \in A_{\mathbb{G}}}$ of projections $\pi_\gamma : K_{\mathbb{G}} \to [-1, 1]$ is a 1-separated sequence of continuous mappings in $C(H_{\mathbb{G}})$. So, we fix $\eta < \gamma$ in $A_{\mathbb{G}}$. Then $\|\pi_\gamma - \pi_\eta\|_{H_{\mathbb{G}}} \geqslant \pi_\gamma(f_\gamma) - \pi_\eta(f_\gamma) = 1$. □

Proposition 8.7. *$K_{\mathbb{G}}$ and $L_{\mathbb{G}}$ are both simplexes.*

Proof. We prove first that $K_{\mathbb{G}}$ is a simplex. To do this, it suffices to check, by Proposition 8.2, that for every condition p

$$\mathcal{D}_p := \{q \in \mathbb{T}\colon \text{ either } q \perp p \text{ or } q \leqslant p \text{ and } \operatorname{conv}_{\mathbb{R}}(F_q) \text{ is a simplex}\}$$

is dense in $\mathbb{T}$. This is the dual counterpart of the fact that a generic Banach space for the basic forcing $\mathbb{P}_{\text{basic}}$ is a Lindenstrauss space and the same proof given in Example 3.10(III) works here.

To check that $L_{\mathbb{G}}$ is a simplex is also easy: In this case, it suffices to check that $\pi_{D_p}^{-1}(\operatorname{conv}_{\mathbb{R}}(H_p \cup \{0\}))$ is a simplex, which is an easy consequence of Proposition 3.4(d). □

Let $S^+ := \{(a_\gamma)_{\gamma < \omega_1} \in \ell_1(\omega_1)\colon\ \|(a_\gamma)_\gamma\|_1 \leqslant 1 \text{ and } a_\gamma \geqslant 0 \text{ for every } \gamma < \omega_1\}$ be the positive part of the unit sphere of $\ell_1(\omega_1)$.

Proposition 8.8. *The mapping $T : S^+ \to L_{\mathbb{G}}$, $T((a_\gamma)_\gamma) = \sum_{\gamma < \omega_1} a_\gamma h_\gamma^{(\mathbb{G})}$ is an affine homeomorphism. In particular,*

$$\operatorname{Ext}(L_{\mathbb{G}}) = H_{\mathbb{G}} \tag{143}$$

and so $L_{\mathbb{G}}$ is a Bauer simplex.

Proof. The first part is proved in Theorem 3.17(1). The equality in (143) is a consequence of the fact that $\operatorname{Ext}(S^+) = \{u_\gamma\}_{\gamma < \omega_1} \cup \{0\}$. The last part of the statement follows from Proposition 8.6. □

Proposition 8.9. *The simplex $K_{\mathbb{G}}$ is a Poulsen simplex.*

Proof. This is essentially the dual version of the fact that a generic Banach space of the basic forcing $\mathbb{P}_{\mathrm{basic}}$ is a Gurarij space. Given a condition p and $f \in \mathrm{conv}_{\mathbb{Q}}(F_p)$ we define $\mathcal{D}_{p,f}$ as the set of all conditions $q \in \mathbb{T}$ such that:

$$\text{Either } q \perp p, \text{ or } q \leqslant p \text{ and there is } g \in \mathrm{Ext}\big(\mathrm{conv}_{\mathbb{R}}(F_q)\big) \text{ such that } g \restriction D_p = f. \tag{144}$$

Claim 8.9.1. *$\mathcal{D}_{p,f}$ is dense.*

Proof. Fix p and f as in the hypothesis, and $q \in \mathbb{T}$. Without loss of generality we assume that $q = p$. Let $D_p < \gamma$. Define $r = (D_r, F_r, H_r)$ as follows. $D_r = D_p \cup \{\gamma\}$, $F_r = F_p \cup \{f + u_\gamma, u_\gamma\}$, $h_\eta^{(r)} = h_\eta^{(p)}$ if $\eta \in D_p$ and $h_\gamma^{(r)} = u_\gamma$. Then $r \leqslant p = q$ and $q \in \mathcal{D}_{p,f}$. $\square$

We now use the claim to prove that $\mathrm{Ext}(K_{\mathbb{G}})$ is dense in $K_{\mathbb{G}}$: Let $h \in K_{\mathbb{G}}$, $s \subseteq \omega_1$ be finite and $\varepsilon > 0$. Let $p \in \mathbb{G}$ be such that $s \subseteq D_p$. Since $h \restriction D_p \in \mathrm{conv}_{\mathbb{R}}(F_p)$, we can find $f \in \mathrm{conv}_{\mathbb{Q}}(F_p)$ such that $\|f - h\|_\infty < \varepsilon$. Since $\mathcal{D}_{p,f}$ is dense and $\mathbb{G}$ is a generic filter, there is $q \in \mathbb{G}$ such that $q \leqslant p$ and there is $g \in \mathrm{Ext}(\mathrm{conv}_{\mathbb{R}}(F_q))$ with $g \restriction D_p = f$. Let now g_0 be an extremal point of $K_{\mathbb{G}}$ extending g. Now

$$\|g_0 \restriction s - h \restriction s\|_\infty = \|f \restriction s - h \restriction s\|_\infty < \varepsilon. \qquad \square \tag{145}$$

Theorem 8.10. *For $K = H_{\mathbb{G}}, L_{\mathbb{G}}, K_{\mathbb{G}}$, the space $P(K)$ of probability measures on K is hereditarily separable in all finite powers.*

The proof is split in several steps.

Lemma 8.11. *Let $(t_\alpha)_{\alpha<\omega_1}$ be an uncountable sequence of finite sets of ω_1, each one of size k, let $t_\alpha = \{\xi_i^{(\alpha)}\}_{i<k}$ be the increasing enumeration of t_α. Then for every integer n there are $\alpha < \omega_1$ and $s \subseteq \omega_1$ with $\alpha < s$ and with $\#(s) = n$ such that*

(∗) *for every $f \in \mathrm{Ext}(K_{\mathbb{G}} \restriction (t_\alpha \cup \bigcup_{\beta \in s} t_\beta))$ there is $\beta(f) \in s$ such that for every $\beta \in s \setminus \{\beta(f)\}$ and every $i < k$ one has that $(f)_{\xi_i^{(\alpha)}} = (f)_{\xi_i^{(\beta)}}$.*

Proof. We shall use Theorem 3.36 for $\mathbb{T}$. Given a Δ-system $(p_i, t_i)_{i<n+1}$ of conditions of $\mathbb{T}$ of type $u = (N, F, H, t)$, $\#t \subseteq N$, $\#t = k$ and root R, there is an amalgamation p such that

(∗∗) for every $f \in F_p$ there is $\beta(f) \in s$ such that for every $\beta \in s \setminus \{\beta(f)\}$ and every $i < k$ one has that $(f)_{\xi_i^{(\alpha)}} = (f)_{\xi_i^{(\beta)}}$.

For each $i \leqslant n$ let $\theta_i : N \to D_i$ be the unique order-preserving bijection. We define $p = (\bigcup_{i \leqslant n} D_i, F_p, H_p)$ where F_p is formed by:

(a) Elements of the form $\bigvee_{i \leqslant n} \theta_i(g)$ for every type $g \in F$.
(b) For every $1 \leqslant i \leqslant n$ and every $\eta \in D_i \setminus R$, F_p contains

$$h_\eta^{(p_i)}.$$

For each $i \leqslant n$ and $\gamma \in D_i$, let $g \in F$ denote the type of $h_\gamma^{(p_i)}$. We define

$$h_\gamma^{(p)} := \begin{cases} \bigvee_{j \leqslant n} \theta_j(g) & \text{if } i = 0, \\ h_\gamma^{(p_i)} & \text{if } i > 0. \end{cases}$$

It is easy to see that $p \leqslant p_i$ for every $i \leqslant n$, and that $(**)$ holds for p. $\square$

Definition 8.12. Recall that given a sequence $\vec{X} := (X_i)_{i<n}$ of sets one defines

$$i(\vec{X}) := \max\Big\{\#(I)\colon\ I \subseteq n \text{ and } \bigcap_{i \in I} X_i \neq \emptyset\Big\}.$$

The *intersection number* $I(\mathcal{F})$ of a family $\mathcal{F}$ of sets is defined by

$$I(\mathcal{F}) := \min\Big\{\frac{i(\vec{X})}{n}\colon\ \vec{X} \in \mathcal{F}^n,\ n \in \mathbb{N}\Big\}.$$

Note the following easy fact about this notion.

Proposition 8.13. *Let μ be a probability measure on $K_{\mathbb{G}}$, and let $\mathcal{F}$ be a finite family of measurable subsets of $K_{\mathbb{G}}$, each $X \in \mathcal{F}$ with $\mu(X) \geqslant \varepsilon$. Then $I(\mathcal{F}) \geqslant \varepsilon \cdot \#(\mathcal{F})$.* $\square$

Proof of Theorem 8.10. We have to check that $P(K_{\mathbb{G}})^n$ is hereditarily separable for all $n \in \mathbb{N}$. We only give the details of the case $n = 1$ because the general case is done in a very similar way. For each $\alpha < \omega_1$, let $\pi_\alpha \in C(K_{\mathbb{G}})$ be the αth-coordinate function, i.e. $\pi_\alpha(x) = (x)_\alpha$ for every $x \in K_{\mathbb{G}}$. Let $\mathcal{A}$ be the algebra generated by the projections π_α, and the constant function $\chi_{K_{\mathbb{G}}}$. Since clearly $\{\pi_\alpha\}_{\alpha<\omega_1}$ separates the points of $K_{\mathbb{G}}$, the Stone–Weierstrass Theorem gives that $\mathcal{A}$ is dense in $C(K_{\mathbb{G}})$. This implies that for a given positive measure μ in $K_{\mathbb{G}}$ the basic open sets

$$U(\mu, \pi_{\alpha_0}, \dots, \pi_{\alpha_k}, \varepsilon) = \Big\{\bar{\mu} \in P(K_{\mathbb{G}})\colon\ \max_{i \leqslant k}\Big|\int \pi_{\alpha_i}\, d(\mu) - \int \pi_{\alpha_i}\, d(\bar{\mu})\Big| < \varepsilon\Big\}$$

when running $\alpha_0 < \cdots < \alpha_k$ and $\varepsilon > 0$ form an open neighborhood basis in μ. Consequently, we have to prove that if $(\mu_\alpha, t_\alpha)_{\alpha<\omega_1}$ is a sequence of pairs of measures μ_α and finite subsets $t_\alpha \subseteq \omega_1$, and if $\varepsilon > 0$, then there is some $\alpha < \beta$ such that

$$\Big|\int \pi_\xi\, d(\mu_\alpha) - \int \pi_\xi\, d(\mu_\beta)\Big| < \varepsilon \quad \text{for every } \xi \in t_\beta. \tag{146}$$

We fix then a such sequence $(\mu_\alpha, t_\alpha)_{\alpha<\omega_1}$ and $\varepsilon > 0$. We may assume, by going to a subsequence if needed and decreasing $\varepsilon > 0$, that:

(a) $\mu_\alpha(K_{\mathbb{G}}) = 1$ for all $\alpha < \omega_1$.
(b) There is an integer k such that $k = \#(t_\alpha)$ for every $\alpha < \omega_1$.

(c) For every $\alpha < \beta < \omega_1$ and every $i < k$ one has that

$$\int \pi_{\xi_i^{(\alpha)}}\, d(\mu_\alpha) = \int \pi_{\xi_i^{(\beta)}}\, d(\mu_\beta), \tag{147}$$

where for a given $\alpha < \omega_1$, $\{\xi_i^{(\alpha)}\}_{i<k}$ is the increasing enumeration of t_α.

Now we apply Lemma 8.11 to $(t_\alpha)_{\alpha<\omega_1}$ and $n > 4/\varepsilon$ to find $\alpha < s$ with $\#(s) = n$ such that the corresponding property $(*)$ in Lemma 8.11 holds. Let us set $K := K_{\mathbb{G}} \upharpoonright (t_\alpha \cup \bigcup_{\beta\in s} t_\beta)$, and $F = \mathrm{Ext}(K)$. For each $f \in K$ we write $f = \sum_{g\in F} a_g^{(f)} \cdot g$. Given $\beta \in s$ we define

$$E_\beta := \left\{ f \in K\colon \sum_{\beta(g)=\beta} a_g^{(f)} > \frac{\varepsilon}{4} \right\}, \qquad G_\beta := K \setminus E_\beta.$$

Let $\mathcal{F} := \{E_\beta\colon\ \beta \in s\}$. We estimate now an upper bound for the intersection number of any $\mathcal{F}_0 \subseteq \mathcal{F}$. Suppose that $J \subseteq s$ is such that $\bigcap_{\beta\in J} E_\beta \neq \emptyset$, and let $f \in \bigcap_{\beta\in J} E_\beta$. Then, if $\#(J) \geqslant 4/\varepsilon$, then there are $\beta_0 < \cdots < \beta_{m-1}$ in s such that

$$\sum_{\beta(g)=\beta_i} a_g^{(f)} > \frac{\varepsilon}{4}, \quad \text{for every } i < m,$$

where m is the entire part of $4/\varepsilon$. Since $g \in F \mapsto \beta(g)$ is a mapping, it follows that

$$\sum_{g\in F} a_g^{(f)} \geqslant \sum_{i<m} \sum_{\beta(g)=\beta_i} a_g^{(f)} > m\frac{\varepsilon}{4} \geqslant 1,$$

and this is impossible. This means that $\#(J) < 4/\varepsilon$. Hence $I(\mathcal{F}_0) < \frac{4}{\varepsilon\cdot\#(\mathcal{F}_0)}$. Applying this to

$$\mathcal{F}_\varepsilon := \left\{ \beta \in s\colon\ \mu_\alpha(E_\beta) \geqslant \frac{\varepsilon}{4} \right\}$$

one obtains that $I(\mathcal{F}_\varepsilon) \leqslant \frac{4}{\varepsilon\cdot\#(\mathcal{F}_\varepsilon)}$. On the other hand, by Proposition 8.13, $I(\mathcal{F}_\varepsilon) \geqslant \frac{\varepsilon}{4}\#(\mathcal{F}_\varepsilon)$. Putting this information together, it follows that $\#(\mathcal{F}_\varepsilon) \leqslant \frac{\varepsilon}{4}$. Now we use that $n > 4/\varepsilon$ to find $\bar\beta \in s$ such that $\mu_\alpha(E_{\bar\beta}) < \frac{\varepsilon}{4}$. We claim that

$$\left| \int \pi_{\xi_i^{(\bar\beta)}}\, d(\mu_\alpha) - \int \pi_{\xi_i^{(\bar\beta)}}\, d(\mu_{\bar\beta}) \right| < \varepsilon \quad \text{for every } i < k: \tag{148}$$

So, we fix $i_0 < k$.

Claim 8.13.1. *If $f \in G_{\bar\beta}$ then $|(f)_{\xi_{i_0}^{(\bar\beta)}} - (f)_{\xi_{i_0}^{(\alpha)}}| < \frac{\varepsilon}{2}$.*

Proof.

$$
\begin{aligned}
\left|(f)_{\xi_{i_0}^{(\bar\beta)}} - (f)_{\xi_{i_0}^{(\alpha)}}\right| &\leqslant \left|\sum_{g\in F,\beta(g)=\bar\beta} a_g^{(f)}\bigl((g)_{\xi_{i_0}^{(\bar\beta)}} - (g)_{\xi_{i_0}^{(\alpha)}}\bigr)\right| + \left|\sum_{g\in F,\beta(g)\neq\bar\beta} a_g^{(f)}\bigl((g)_{\xi_{i_0}^{(\bar\beta)}} - (g)_{\xi_{i_0}^{(\alpha)}}\bigr)\right| \\
&\leqslant \sum_{g\in F,\beta(g)=\bar\beta} a_g^{(f)}\left|(g)_{\xi_{i_0}^{(\bar\beta)}} - (g)_{\xi_{i_0}^{(\alpha)}}\right| \leqslant \frac{\varepsilon}{4}2 = \frac{\varepsilon}{2}. \qquad \square
\end{aligned}
$$

We estimate now:

$$
\begin{aligned}
\left|\int \pi_{\xi_{i_0}^{(\bar\beta)}}\, d(\mu_{\bar\beta}) - \int \pi_{\xi_{i_0}^{(\bar\beta)}}\, d(\mu_\alpha)\right| &= \left|\int \pi_{\xi_{i_0}^{(\bar\beta)}}\, d(\mu_{\bar\beta}) - \int \pi_{\xi_{i_0}^{(\alpha)}}\, d(\mu_\alpha)\right| \\
&= \left|\int (\pi_{\xi_{i_0}^{(\bar\beta)}} - \pi_{\xi_{i_0}^{(\alpha)}})\, d(\mu_\alpha)\right| \\
&\leqslant \left|\int_{E_{\bar\beta}} (\pi_{\xi_{i_0}^{(\bar\beta)}} - \pi_{\xi_{i_0}^{(\alpha)}})\, d(\mu_\alpha)\right| + \int_{G_{\bar\beta}} \left|\pi_{\xi_{i_0}^{(\bar\beta)}} - \pi_{\xi_{i_0}^{(\alpha)}}\right| d(\mu_\alpha) \\
&< 2\cdot \mu_\alpha(E_{(i_0,\bar\beta)}) + \mu_\alpha(G_{(i_0,\bar\beta)})\frac{\varepsilon}{2} < \varepsilon. \qquad \square
\end{aligned}
$$

Corollary 8.14. *The weak topologies of the function spaces $C(H_{\mathbb{G}})$, $C(K_{\mathbb{G}})$ and $C(L_{\mathbb{G}})$ are all hereditarily Lindelöf in their finite powers.* □

Remark 8.15. The same proof gives that if $X_{\mathbb{G}}$ is a generic space for the basic forcing notion $\mathbb{P}_{\text{basic}}$, then $C(B_{(X_{\mathbb{G}})^*})$ is hereditarily Lindelöf in all its powers.

8.1. Support sets in $C(K)$-spaces and perfect Choquet simplexes

We first present a generic zero-dimensional compactum $K_{\mathbb{G}}^0$ whose space $P(K_{\mathbb{G}}^0)$ of Radon probability measures is hereditarily separable in all finite powers and whose function space $C(K_{\mathbb{G}}^0)$ does not have support sets. Indeed this is the generic compactum introduced in [2]. We recall its definition.

Definition 8.16. Let $\mathbb{T}_0$ be the forcing notion whose conditions are pairs $p=(D_p,F_q)$ where D_p is a finite subset of ω_1 and where $F_p\subseteq c_{00}(D_p,\{0,1\})$ is a set with the property that

$$
\begin{gathered}
\text{for every } \gamma\in D_p \text{ there exist } f,g\in F_p \text{ such that} \\
f\restriction\gamma = g\restriction\gamma \text{ and } (f)_\gamma = 1 \text{ and } (g)_\gamma = 0.
\end{gathered} \tag{149}
$$

We order $\mathbb{T}_0$ by letting $p\leqslant q$ if $D_p\supseteq D_q$ and $F_p\restriction D_q = F_q$.

It is easy to see that $\mathbb{T}_0$ has the Shanin property and that $D_{\mathbb{G}}=\omega_1$ for every generic filter $\mathbb{G}$ of $\mathbb{T}_0$. Fix such generic filter $\mathbb{G}$ and consider the following compact subset of $[0,1]^{\omega_1}$,

$$K^0_{\mathbb{G}} := \bigcap_{p\in\mathbb{G}} \pi^{-1}_{D_p}(F_p).$$

The following fact easily follows from the definition.

Proposition 8.17. *The compact space $K^0_{\mathbb{G}}$ is non-metrizable and zero-dimensional.* □

Theorem 8.18. *The function space $C(K^0_{\mathbb{G}})$ does not have support sets and the space $P(K^0_{\mathbb{G}})$ of all Radon probability measures on $K^0_{\mathbb{G}}$ is hereditarily separable in all finite powers.*

Proof. We first prove that the function space $C(K^0_{\mathbb{G}})$ has no support sets. We follow the lines of the proof of Theorem 7.3. Going towards a contradiction, we suppose there is an uncountable semi-biorthogonal system. Then a simple approximation argument gives an uncountable sequence $(x_\alpha)_{\alpha<\omega_1}$ of polynomials with rational coefficients, and some integer $\bar{k}$ such that for every $k \geqslant \bar{k}$ and every finite sets $A < \alpha < B$, $\#A = 2k$ and $\#B = k$ one has that

$$\left\| -\sum_{\alpha\in A} x_\alpha + kx_{\bar{\alpha}} + \sum_{\alpha\in B} x_\alpha \right\| \geqslant 3. \tag{150}$$

We prove that this is not possible. Since each x_α is a polynomial, it is a Lipschitz function, so going to an uncountable sequence and after re-enumeration if needed, we may assume that each x_α is k-Lipschitz for some fixed $k \geqslant \bar{k}$.

Let $\mathfrak{P}(v_0,\dots,v_{3k})$ be the configuration: Either $\|v_i\| \neq 1$ for some $i \leqslant 3k$ or else

$$\left\| -\sum_{i<2k} v_i + kv_{2k} + \sum_{i=2k+1}^{3k} v_i \right\| \leqslant 2. \tag{151}$$

Now it suffices to prove that $\mathfrak{P}$ is unavoidable, and we use the Forcing Theorem 3.36. So, we fix a Δ-system $(p_i, v_i)_{i\leqslant 3k}$ of type $t = (N, F, v)$ and root R. Let $\theta_i : N \to D_{p_i}$ be the corresponding order-preserving bijection for every $i \leqslant 3k$, and for each $\gamma \in N$, we fix $f_\gamma, g_\gamma \in F$ such that $(f_\gamma) \upharpoonright \gamma = (g_\gamma) \upharpoonright \gamma$ and $(f_\gamma)_\gamma = 1$ and $(g_\gamma)_\gamma = 0$. Without loss of generality we assume that $\|v\| = 1$. We define the amalgamation $p = (\bigcup_{i\leqslant 3k} D_{p_i}, F_p)$, where the elements of F_p are:

(a) $\bigvee_{i\leqslant 3k} \theta_i(f)$ for every $f \in F$.
(b) $\bigvee_{j<i} \theta_j(f_\gamma) \vee \theta_i(g_\gamma) \vee \bigvee_{i<j\leqslant 3k} \theta_j(f_\gamma)$ for every $i \neq 2k$, and $\gamma \in N \setminus |R|$.
(c) $f^{(p)}_\gamma := \bigvee_{i<k}(\theta_{2i}(f_\gamma) \vee \theta_{2i+1}(g_\gamma)) \vee \theta_{2k}(f_\gamma) \vee \bigvee_{2k<i\leqslant 3k} \theta_i(g_\gamma)$ and $g^{(p)}_\gamma := \bigvee_{i<k}(\theta_{2i}(f_\gamma) \vee \theta_{2i+1}(g_\gamma)) \vee \theta_{2k}(g_\gamma) \vee \bigvee_{2k<i\leqslant 3k} \theta_i(f_\gamma)$ for every $\gamma \in N \setminus |R|$.

We check (149) for p: Fix $\gamma \in D_{p_i}$ for some $i \leqslant 3k$. Let $\eta \in N$ be such that $\theta_i(\eta) = \gamma$. Suppose first that $\gamma \in R$. Then $\bigvee_{j\leqslant 3k} \theta_j(f_\eta)$ and $\bigvee_{j\leqslant 3k} \theta_j(g_\eta)$ do the job. Suppose now that $\gamma \notin R$ and $i \neq 2k$. Then $\bigvee_{j\leqslant 3k} \theta_j(f_\eta)$ and $\bigvee_{j<i} \theta_j(f_\gamma) \vee \theta_i(g_\gamma) \vee \bigvee_{i<j\leqslant 3k} \theta_j(f_\gamma)$ work. Finally, if $\gamma \notin R$ and $i = 2k$ then $f^{(p)}_\gamma$ and $g^{(p)}_\gamma$ are the desired pair.

Now it is routine to check that the configuration $\mathfrak{P}(v_i)_{i\leqslant 3k}$ holds in p.

The fact that $P(K^0_{\mathbb{G}})$ is hereditarily separable follows from the fact that we have the analogue of Lemma 8.11 for $\mathbb{T}_0$. This is easily shown following the lines of the proof of that lemma. □

Already from the early papers about the Rolewicz problem we learn that if a $C(K)$-space admits no support sets then the corresponding compactum must be both hereditarily separable and hereditarily Lindelöf (see, for example, [19]). This shows that there is a striking difference between the forcing constructions of $\mathbb{T}$ and $\mathbb{T}_0$ described above since neither of the compacta $H_{\mathbb{G}}$, $K_{\mathbb{G}}$ and $L_{\mathbb{G}}$ given by $\mathbb{T}$ can be hereditarily Lindelöf. The key difference is hidden in the way the non-metrizability of the corresponding compacta is imposed, i.e., in the conditions of Definition 8.3 and the condition (149) of Definition 8.16. In fact there is even a structural reason for the difference since it is easily seen that every $C(K)$ space over a symmetric convex compactum K does have a support sets and that no Bauer simplex can be hereditarily Lindelöf unless it is metrizable (or, more generally, a compact convex set whose sets of extremal points is both Lindelöf and G_δ is metrizable; see, for example, [34]). We now show that the natural convex version of the forcing notion given in Definition 8.16 does give us a non-metrizable perfect[7] convex compactum.

Definition 8.19. Let $\mathbb{T}_C$ be the forcing notion with conditions the pairs $p = (D_p, F_q)$, where D_p is a finite subset of ω_1, and where $F_p \subseteq c_{00}(D_p, \mathbb{Q} \cap [-1, 1])$ is a finite set with property that

$$\text{for every } \gamma \in D_p \text{ there exist } f, g \in F_p \text{ such that}$$
$$f \restriction \gamma = g \restriction \gamma \text{ and } (f)_\gamma = 1 \text{ and } (g)_\gamma = 0. \tag{152}$$

We order $\mathbb{T}_C$ by letting $p \leqslant q$ if $D_p \supseteq D_q$ and

$$F_q \subseteq F_p \restriction D_q \subseteq \mathrm{conv}_{\mathbb{Q}}(F_q).$$

Then $\mathbb{T}_C$ has the Shanin property and $D_{\mathbb{G}} = \omega_1$ for every generic filter $\mathbb{G}$ of $\mathbb{T}_C$. If for a generic filter $\mathbb{G}$ we define the following compact subset of $[0, 1]^{\omega_1}$,

$$K_{\mathbb{G}}^C := \bigcap_{p \in \mathbb{G}} \pi_{D_p}^{-1}\big(\mathrm{conv}_{\mathbb{R}}(F_p)\big),$$

we get a compact convex non-metrizable compactum.

Theorem 8.20. *The compact space $K_{\mathbb{G}}^C$ is a perfect non-metrizable Poulsen simplex. Moreover, $P(K_{\mathbb{G}}^C)$ is hereditarily separable in all finite powers.*

Proof. We do not give a proof that $K_{\mathbb{G}}^C$ is a Poulsen simplex and that $P(K_{\mathbb{G}}^C)$ is hereditarily separable (HS) in all finite powers as it is very similar to corresponding facts for $K_{\mathbb{G}}$ above. Instead, we concentrate in proving that $K_{\mathbb{G}}^C$ is perfect. Recall that for $x \in c_{00}(\omega_1, \mathbb{Q} \cap [-1, 1])$ and $\varepsilon > 0$, we write $[x]_\varepsilon$ to denote

$$[x]_\varepsilon := \big\{ f \in K_{\mathbb{G}}^C \colon \big|(f)_\gamma - (x)_\gamma\big| < \varepsilon \text{ for every } \gamma \in \operatorname{supp} x \big\}.$$

We shall prove the following combinatorial property which easily implies (HL).

[7] We recall that a topological space X is *perfect* if every closed subset of X is G_δ. Clearly, a compact space is perfect if and only if it is hereditarily Lindelöf.

Claim 8.20.1. *For every sequence $(x_\alpha)_{\alpha<\omega_1}$ in $c_{00}(\omega_1, \mathbb{Q}\cap[-1,1])$ and every $\varepsilon>0$ there is a positive integer n and a sequence $\alpha_0<\cdots<\alpha_n$ of countable ordinals such that*

$$[x_{\alpha_n}]_{\frac{\varepsilon}{2}} \subseteq \bigcup_{i<n}[x_i]_\varepsilon. \tag{153}$$

Proof. It suffices to prove that given an uncountable sequence $(p_\alpha, x_\alpha)_{\alpha<\omega_1}$ of pairs $(p_\alpha, x_\alpha)\in \mathbb{T}\times c_{00}(\omega_1, \mathbb{Q}\cap[-1,1])$ such that $x_\alpha\in c_{00}(D_{p_\alpha})$ there are $\alpha_0<\cdots<\alpha_n$ and $p\leqslant p_{\alpha_i}$ such that

$$[x_{\alpha_n}]_{\frac{\varepsilon}{2}} \restriction \mathrm{conv}_{\mathbb{R}}(F_p) \subseteq \bigcup_{i<n}[x_i]_\varepsilon \restriction \mathrm{conv}_{\mathbb{R}}(F_p), \tag{154}$$

where $[x]_\varepsilon \restriction \mathrm{conv}_{\mathbb{R}}(F_p)=\{f\in \mathrm{conv}_{\mathbb{R}}(F_p)\colon \max_{\gamma\in\operatorname{supp} x}|(f)_\gamma-(x)_\gamma|<\varepsilon\}$, for a given $x\in c_{00}(\omega_1)$. By going to an uncountable subsequence we may assume that $(p_\alpha, x_\alpha)_{\alpha<\omega_1}$ is a Δ-system with root R and type $t:=(N,F,x)$. Let $\bar{N}=N-\#R$, and l be an integer such that $l\varepsilon\geqslant 6$, and let n be such that

$$\left(\frac{1}{l}\right)^{\bar{N}} > \frac{l}{n}. \tag{155}$$

For each $i\leqslant n$ let $\theta_i: N\to D_i$ be the corresponding order-preserving bijection, and for each $\gamma\in\bar{N}$, let $f_\gamma, g_\gamma\in F$ be such that $f_\gamma\restriction\gamma = g_\gamma\restriction\gamma$ and $(f_\gamma)_\gamma=1$ and $(g_\gamma)_\gamma=0$. Let $p=(\bigcup_{i\leqslant n}D_i, F_p)$ be the following amalgamation of $(p_i)_{i\leqslant n}$. The set F_p contains:

(1) $\bigvee_{i\leqslant n}\theta_i(f)$ for every $f\in F$.
(2) $\bigvee_{j\leqslant n, j\neq i}\theta_j(f_\gamma)\vee\theta_i(g_\gamma)$ for every $i<n$ and every $\gamma\in N\setminus\#R$.
(3) For each $\gamma\in\bar{N}$, choose a partition $\{A_\gamma^{(0)},\dots,A_\gamma^{(l-1)}\}$ of n by sets of equal size that are stochastically independent, i.e.,

$$\#\left(\bigcap_{\gamma\in\bar{N}}A_\gamma^{(\sigma(\gamma))}\right)=n\cdot\left(\frac{1}{l}\right)^{\bar{N}}, \quad \text{for every } \sigma\in l^{\bar{N}}. \tag{156}$$

For each $\gamma\in\bar{N}$ and each $k<l$, let

$$e_{\gamma,k}:=\frac{k}{l}f_\gamma+\frac{l-k}{l}g_\gamma\in \mathrm{conv}_{\mathbb{Q}}(F). \tag{157}$$

Now for every $\gamma\in\bar{N}$, the set F_p contains

$$f_\gamma^{(p)}:=\bigvee_{k<l}\bigvee_{i\in A_\gamma^{(k)}}\theta_i(e_{\gamma,k})\vee\theta_n(f_\gamma) \quad \text{and} \tag{158}$$

$$g_\gamma^{(p)}:=\bigvee_{k<l}\bigvee_{i\in A_\gamma^{(k)}}\theta_i(e_{\gamma,k})\vee\theta_n(g_\gamma). \tag{159}$$

It should be clear that $p \in \mathbb{T}$ and that $p \leqslant p_i$ for all $i \leqslant n$. Finally, let us prove that p is the required amalgamation. We say that $h \in F_p$ has type (i) if it is as in (i) above in the definition of F_p, for $i \leqslant 3$. Now for a given h of type (2), let $i(h)$ denote the corresponding integer $i < n$ such that $h = \bigvee_{j \leqslant n, j \neq i} \theta_j(f_\gamma) \vee \theta_i(g_\gamma)$ for some $\gamma \in \bar{N}$. We say that such h is of type $(2, i(h))$. Observe that this a disjoint partition $F_p^{(1)}, \{F_p^{(2,i)}\}_{i<n}, F_p^{(3)}$ of F_p defined by the types.

Now fix $e \in \mathrm{conv}_{\mathbb{R}}(F_p)$, and write $e = \sum_{h \in F_p} a_h h$, convex combination. Let

$$\begin{aligned} b_j &:= \sum \{a_h\colon h \text{ has type } (j)\}, \quad \text{for } j = 1, 3, \\ b_{2,i} &:= \sum \{a_h\colon h \text{ has type } (2, i)\}, \quad \text{for } i < n. \end{aligned}$$

Then

$$\sum \{a_h h\colon h \text{ of type } (3)\} = \sum \{b_\gamma f_\gamma^{(p)} + c_\gamma g_\gamma^{(p)}\colon \gamma \in \bar{N}\}, \tag{160}$$

where $b_\gamma := a_{f_\gamma^{(p)}}$ and $c_\gamma := a_{g_\gamma^{(p)}}$ for each $\gamma \in \bar{N}$. Let $\bar{\bar{N}}$ be the set of all $\gamma \in \bar{N}$ such that $b_\gamma + c_\gamma \neq 0$, and for each $\gamma \in \bar{\bar{N}}$, let $k_\gamma < l$ be such that

$$\bar{b}_\gamma f_\gamma + \bar{c}_\gamma g_\gamma \in [e_{\gamma, k_\gamma}, e_{\gamma, k_\gamma + 1}[, \tag{161}$$

where $\bar{b}_\gamma = b_\gamma / (b_\gamma + c_\gamma)$ and $\bar{c}_\gamma = c_\gamma / (b_\gamma + c_\gamma)$. We declare in addition that $k_\gamma := 0$ for every $\gamma \in \bar{N} \setminus \bar{\bar{N}}$. Now, because of (156) applied to $\sigma = (k_\gamma)_{\gamma \in \bar{N}}$, we get that

$$A := \bigcap_{\gamma \in \bar{N}} A_\gamma^{(k_\gamma)} \text{ has cardinality at least } l + 1. \tag{162}$$

For each $i < n$, let $d_i := \sum \{a_h\colon h \text{ has type } (2, i)\}$. Notice that $\sum_{i<n} d_i \leqslant 1$, so in particular, by (162), there must be $i_0 \in A$ such that $d_{i_0} \leqslant 1/l \leqslant \varepsilon/6$. We claim that

$$\left|(e)_{\theta_{i_0}(\gamma)} - (e)_{\theta_n(\gamma)}\right| \leqslant \frac{\varepsilon}{2} \quad \text{for every } \gamma \in N. \tag{163}$$

If in particular, $e \in [x_n]_{\varepsilon/2}$, then $e \in [x_{i_0}]_\varepsilon$. To see (163), fix $\gamma \in N$. If $\gamma \in \#R$ then the result is trivial. So, suppose that $\gamma \in \bar{N}$. Then,

$$\begin{aligned} (e)_{\theta_{i_0}(\gamma)} &= \sum \{a_h(h)_{\theta_{i_0}(\gamma)}\colon h \text{ of type } (1)\} + \sum_{i<n, i \neq i_0} \sum \{a_h(h)_{\theta_{i_0}(\gamma)}\colon h \text{ of type } (2, i)\} \\ &\quad + \sum \{a_h(h)_{\theta_{i_0}(\gamma)}\colon h \text{ of type } (2, i_0)\} + \sum \{a_h(h)_{\theta_{i_0}(\gamma)}\colon h \text{ of type } (3)\} \\ &= \sum \{a_h(h)_{\theta_n(\gamma)}\colon h \text{ of type } (1)\} + \sum_{i<n, i \neq i_0} \sum \{a_h(h)_{\theta_n(\gamma)}\colon h \text{ of type } (2, i)\} \\ &\quad + \sum \{a_h(h)_{\theta_{i_0}(\gamma)}\colon h \text{ of type } (2, i_0)\} + \sum \{a_h(h)_{\theta_{i_0}(\gamma)}\colon h \text{ of type } (3)\}. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|(e)_{\theta_{i_0}(\gamma)} - (e)_{\theta_n(\gamma)}\right| \\
&\quad = \bigg| \sum\big\{a_h(h)_{\theta_{i_0}(\gamma)}\colon h \text{ of type } (2, i_0)\big\} - \sum\big\{a_h(h)_{\theta_n(\gamma)}\colon h \text{ of type } (2, i_0)\big\} \\
&\qquad + \sum_{\eta\in\bar N}(b_\eta + c_\eta)\big(\bar b_\eta f_\eta^{(p)} + \bar c_\eta g_\eta^{(p)}\big)_{\theta_{i_0}(\gamma)} - \sum_{\eta\in\bar N}(b_\eta + c_\eta)\big(\bar b_\eta f_\eta^{(p)} + \bar c_\eta g_\eta^{(p)}\big)_{\theta_n(\gamma)}\bigg| \\
&\quad \leqslant \frac{\varepsilon}{3} + \max_{\eta\in\bar N}\big|\big(\bar b_\eta f_\eta^{(p)} + \bar c_\eta g_\eta^{(p)}\big)_{\theta_{i_0}(\gamma)} - \big(\bar b_\eta f_\eta^{(p)} + \bar c_\eta g_\eta^{(p)}\big)_{\theta_n(\gamma)}\big|. \qquad (164)
\end{aligned}$$

On the other hand, for every $\eta \in \bar N$ we have that

$$\big(\bar b_\eta f_\eta^{(p)} + \bar c_\eta g_\eta^{(p)}\big)_{\theta_{i_0}(\gamma)} = (e_{\eta,k_\eta})_\gamma,$$
$$\big(\bar b_\eta f_\eta^{(p)} + \bar c_\eta g_\eta^{(p)}\big)_{\theta_n(\gamma)} = (\bar b_\eta f_\eta + \bar c_\eta g_\eta)_\gamma.$$

So, it follows from (161) and the choice of l that for each $\eta \in \bar N$ we have that

$$\big|\big(\bar b_\eta f_\eta^{(p)} + \bar c_\eta g_\eta^{(p)}\big)_{\theta_{i_0}(\gamma)} - \big(\bar b_\eta f_\eta^{(p)} + \bar c_\eta g_\eta^{(p)}\big)_{\theta_n(\gamma)}\big| \leqslant \frac{1}{l} < \frac{\varepsilon}{6}. \qquad (165)$$

Using (165) in (164), we obtain the desired result in (163). □

The following fact shows that the simplex $K_{\mathbb{G}}^C$ gives answers to two problems about perfect compacta found in the literature (see [29] and [10, Problem (DN)]).

Corollary 8.21. *The space $K_{\mathbb{G}}^C$ is a perfect convex compactum which is not metrizably fibered, or in other words, every continuous map from $K_{\mathbb{G}}^C$ into a metric compactum is constant on a non-metrizable subset.* □

Proof. This is so because continuous maps from $K_{\mathbb{G}}^C$ into metric compacta factor though projection maps $\pi_\alpha : K_{\mathbb{G}}^C \to [-1, 1]^\alpha$ $(\alpha < \omega_1)$ which do not have metrizable fibers by an easy density argument. □

In this context and in connection with the Problem (DN) of [10] the reader should note the general fact that no non-metrizable compact convex set is an at most 2-to-1 continuous pre-image of a metrizable compactum.

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